Monetary Policy and Sentiment Driven Fluctuations

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Abstract

Conventional stabilization policy may not robust to expectation driven fluctuations. When prices are sticky and set based on an endogenous signal of demand, sentiments, i.e. beliefs about aggregate demand, can generate stochastic self-fulfilling rational expectations equilibria. The set of equilibria is endogenous to policy. Targeting inflation strongly can increase the set of non-fundamental equilibria, increasing output volatility while decreasing inflation volatility. Even when the Taylor principle is satisfied, there is indeterminacy. The policymaker can eliminate non-fundamental equilibria by relaxing its response to inflation. However, if payoff-relevant aggregate shocks are present as well, the policymaker faces a more interesting tradeoff. It may not be able to respond optimally, in that it can not stabilize the economy from the effects of fundamental shocks without allowing non-fundamental equilibria to arise.

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1 Introduction

This paper presents a model in which monetary policy can change the set of equilibrium outcomes when incomplete information generates self-fulfilling sunspot equilibria. In the benchmark model, although firm level production is not characterized by complementarities or externalities, multiple rational expectations equilibria may result from the signal extraction problem of firms. Sentiments, or beliefs about aggregate output are a source of extrinsic fluctuations, with implications for intrinsic variables such as output and employment. Moreover the volatility of sentiments can be pinned down by structural parameters. By introducing price rigidity in the form of Calvo price setting, I show how monetary policy can modify the volatility of output and inflation in sentiment driven equilibria through its effect on the optimal response of firms to extrinsic shocks. The result is that indeterminacy may arise even when the Taylor principle is satisfied. These results also hold for the case of wage stickiness and a central bank that targets wage inflation.

The benchmark model of Benhabib et al. (2015) is extended to incorporate rigidities in prices and wages, as well as the interaction between fundamental and extrinsic sources of fluctuations. When productivity shocks are introduced, equilibria multiplicity and amplification in the transmission of shocks to output result from agents responding to an endogenous signal. In this regard, the information structure is similar to Chahrour and Gaballo (2017), which considers consumers who learn from prices, an endogenous signal that combines idiosyncratic and aggregate elements.

The equilibrium considered in this model is similar to that in Angeletos and La'O (2013a), where sentiment driven fluctuations arise from firms' uncertainty regarding demand and consumer uncertainty regarding employment and income. However, a feature of the model that follows is that the distribution of sentiments is pinned down endogenously by structural parameters and corresponds to the self-fulfilling distribution of aggregate output.

In this paper, I consider the ability of monetary policy to stabilize fluctuations that arise from information frictions. The normative effects of policy are also considered in Angeletos and La'O (2010), where sentiment driven fluctuations can arise from lack of common knowledge regarding aggregate real shocks. As the source of fluctuations is information heterogeneity, policy intervention can not improve outcomes.

The topic of this paper is also related to Angeletos and Werning (2006), Hellwig et al. (2006), Angeletos et al. (2007)) which consider models where endogenous policy responses provide information about fundamentals, facilitating coordination and multiplicity.

The rest of the paper is organized as follows. Section (1) presents a simple beauty contest model to illustrate how information frictions may lead to indeterminacy in the aggregate response, while the volatility fluctuations depend on structural parameters. In section (2), the benchmark model is considered, with information asymmetries to generate the same results as the previous section. Section (3) incorporates a form of wage rigidity into the benchmark model in order to analyze the effect of monetary policy on equilibrium outcomes. Section (4) and (5) consider the case of price setting firms and price rigidities a la Calvo. Optimal monetary policy is considered in (6). Section (7) explores the interaction between extrinsic shocks and fundamental (payoff relevant) shocks. Finally, section (8) concludes with possible extensions of the analysis.

2 Information Frictions in a Beauty Contest Model

In this model, multiple equilibria does not rely on non-convexities in technology or preferences, externalities, or randomizations over fundamental equilibria, but on information frictions. To illustrate its role in generating multiple equilibria, consider the first order condition of a beauty contest model, where a continuum of agents indexed by $j \in [0,1]$ take action

$$y_{j,t} = \mathbb{E}[\beta_0 \varepsilon_{j,t} + \beta_1 y_t | I_{j,t}]$$

Agent j's optimal response depends on an idiosyncratic iid shock $\varepsilon_{j,t} \sim N(0, \sigma_\varepsilon^2)$, as well as on the aggregate response of other agents ($y_t = \int_0^1 y_{j,t} dj$). The parameters β_0 and β_1 capture the elasticity of actions to the idiosyncratic shock and the aggregate variable, respectively. If $\beta_1 > 0$, agents face strategic complementarities. If $\beta_1 < 0$, agents face strategic substitutabilities. In the case of incomplete information, agents condition their response on a unique information set, denoted by $I_{j,t}$.

2.1 Complete Information

In the complete information case,

$$y_{j,t} = \beta_0 \varepsilon_{j,t} + \beta_1 y_t$$

Assuming the law of large numbers applies across agents,

$$\begin{aligned} y_t &= \int_0^1 y_{j,t} dj \\ &= \int_0^1 (\beta_0 \varepsilon_{j,t} + \beta_1 y_t) dj \\ &= \beta_1 y_t \end{aligned}$$

In the case of $\beta_1 \neq 1$, the only equilibrium is $y_t = 0$. In the knife-edge case that $\beta_1 = 1$, then multiple equilibria exist and any y_t is a solution. Note that this is not a knife edge case, as for any σ_z^2, there exists a set of parameters such that this is an equilibrium.

2.2 Incomplete Information

In the incomplete information case, agents do not observe $\varepsilon_{j,t}$ and y_t. Instead, they condition their response on an imperfect private signal of these components[2]

$$s_{j,t} = \lambda \varepsilon_{j,t} + (1-\lambda) y_t$$

[2]See appendix (7.1) for an explanation of why, when firms' actions are strategic substitutes, a sentiment driven equilibrium exists only if the private signal contains $\varepsilon_{j,t}$ and z_t in proportions different from the firms' first order condition; i.e. where $\lambda \neq \beta_0$ and $(1-\lambda) \neq \beta_1$.

Assuming $y_t \sim N(0, \sigma_y^2)$, the best response of firm j conditional on its signal, by Bayesian updating:

$$\begin{aligned}
y_{j,t} &= \beta_0 \mathbb{E}[\varepsilon_{j,t}|s_{j,t}] + \beta_1 \mathbb{E}[y_t|s_{j,t}] \\
&= \beta_0 \frac{\lambda\sigma_\varepsilon^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_y^2} s_{j,t} + \beta_1 \frac{(1-\lambda)\sigma_y^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_y^2} s_{j,t} \\
&= \frac{\beta_0\lambda\sigma_\varepsilon^2 + \beta_1(1-\lambda)\sigma_y^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_y^2}(\lambda\varepsilon_{j,t} + (1-\lambda)y_t)
\end{aligned}$$

The aggregate action across agents is:

$$y_t = \int_0^1 y_{j,t} dj = \frac{\beta_0\lambda\sigma_\varepsilon^2 + \beta_1(1-\lambda)\sigma_y^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_y^2}(1-\lambda)y_t \tag{1}$$

which can be decomposed as follows:

$$y_t = \beta_0 \underbrace{\frac{\lambda\sigma_\varepsilon^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_y^2}(1-\lambda)y_t}_{\text{pass-through of } y_t \text{ on } \mathbb{E}[\varepsilon_{j,t}]} + \beta_1 \underbrace{\frac{(1-\lambda)\sigma_y^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_y^2}(1-\lambda)y_t}_{\text{pass-through of } y_t \text{ on } \mathbb{E}[y_t]}$$

In addition to the fundamental equilibrium where $y_t = 0$, there may be a sentiment driven equilibrium, where any y_t coming from a distribution with volatility σ_y^2 is a self-fulfilling equilibrium. Equating the coefficients in (1):

$$\frac{\beta_0\lambda\sigma_\varepsilon^2 + \beta_1(1-\lambda)\sigma_y^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_y^2}(1-\lambda) = 1$$

In this case, the volatility of output in the sentiment driven equilibrium is determined by the parameters of the model $(\beta_0, \beta_1, \lambda)$

$$\sigma_y^2 = \frac{\lambda(1-\lambda)\beta_0 - \lambda^2}{(1-\beta_1)(1-\lambda)^2}\sigma_\varepsilon^2$$

A sentiment driven equilibrium with fluctuations in y_t exists if $\sigma_y^2 > 0$, which requires two conditions:

1. Fluctuations in idiosyncratic shock

$$\sigma_\varepsilon^2 > 0$$

   To see why this condition is necessary, suppose instead that $\sigma_\varepsilon^2 = 0$. Then $\varepsilon_{j,t} = \varepsilon \ \forall \ j$ where ε is constant and known. Then the private signal conveys only information about the aggregate variable, and the equilibrium aggregate response is constant. The action of agent j:

$$y_{j,t} = \beta_0\varepsilon + \beta_1\mathbb{E}_t(y_t|s_{j,t})$$

   Conditional on signal $s_{j,t} = \lambda\varepsilon + (1-\lambda)y_t$, agent j chooses

$$y_{j,t} = \beta_0\varepsilon + \beta_1 y_t$$

Summing across agents, the aggregate action is

$$
\begin{aligned}
y_t &= \beta_0 \varepsilon + \beta_1 y_t \\
&= \frac{\beta_0 \varepsilon}{1 - \beta_1}
\end{aligned}
$$

2. Information frictions are also necessary for a sentiment driven equilibrium. The elasticity the agent's optimal response with respect to the aggregate response of others (β_1) must be commensurate with the sentiment proportion of the private signal ($1-\lambda$).

   - If $\beta_1 < 1$ and $1 - \lambda > \frac{1}{1+\beta_0}$
   - If $\beta_1 > 1$ and $1 - \lambda < \frac{1}{1+\beta_0}$

   Consider the case where agents' actions are strategic substitutes ($\beta_1 < 1$): As the volatility of sentiments increases, agents attribute more of their signal to sentiments. If the optimal response of agents decreases when the aggregate response increases, for a sentiment-driven equilibrium to be possible, the sentiment component of the signal is sufficiently large for enough of y_t to pass through to the aggregate response. If $1-\lambda$ is sufficiently large, then enough of the sentiment component is misattributed to idiosyncratic component such that for any y_t, a self-fulfilling equilibrium exists.[3]

   An alternative way to think about the information friction is to say that for a sentiment driven equilibrium to be possible, the response of agents to the idiosyncratic shock (β_0) must be opposite to the response to the aggregate variable (β_1).

   - If $\beta_1 < 1$ and $\beta_0 > \frac{\lambda}{1-\lambda}$
   - If $\beta_1 > 1$ and $\beta_0 < \frac{\lambda}{1-\lambda}$

   If agents want to respond differently to idiosyncratic shock and to the aggregate variable, but can not distinguish between the two in their signal, then a coordinated over-response or under-response across agents can lead to sentiment-driven equilibria.

The second condition shows how a sentiment driven equilibrium requires a relationship between β_1, λ and σ_y^2. Consequently, if there is a change in the degree of substitutability or complementarity in the agents' actions (β_1), this has implications for the equilibrium volatility of the sentiments. For example, if agents' actions are characterized by more substitutability, then for any y_t to be self-fulfilling, its volatility must decrease in order for agents to attribute less of their signal to the sentiment component and to reduce their best response.

3 Benchmark Model with Price Setting Firms

The first section will consider the micro-foundations of the baseline model (decisions of households, firms, equilibrium condition). The quantity of output in the fundamental equilibrium is derived, followed by the mean level of output in the sentiment driven equilibrium.

[3]Note that if $\lambda = 0$, the private signal reports only y_t. If $\beta_1 \neq 1$, the only equilibrium is $y_t = 0$. Otherwise if $\beta_1 = 1$, any y_t can be an equilibrium. If $\lambda = 1$, the private signal contains only the idiosyncratic shock, and if $\beta_0(1-\lambda) \neq 1$, the unique equilibrium is $y_t = 0$. Otherwise, if $\beta_0(1-\lambda) \neq 1$, any y_t can be an equilibrium.

In addition, the mechanism behind a self-fulfilling equilibrium with sentiments (how sentiments lead to fluctuations in output) will be described. The second section will consider the impact of monetary policy on the volatility of output in the sentiment-driven equilibrium. In the benchmark model (Benhabib et al. (2015)), there is a representative household and a continuum of monopolistic intermediate goods producers indexed by $j \in [0, 1]$. Households supply labor and form *demand schedules* for differentiated goods conditional on shocks that have not yet been realized. The key friction is that intermediate goods firms must set prices first and commit to meeting demand at the announced price, based on an imperfect signal of the aggregate demand and firm level demand.

After prices are set, the goods market opens, demand is realized, and production adjust to meet demand at the announced price. The firms' signal extraction problem can lead to multiple equilibria and endogenous fluctuations in aggregate output.

3.1 Households

The representative household's problem is[4]

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} + \Psi(1 - N_t) \right)$$

subject to

$$C_t \equiv \left[\int \epsilon_{j,t}^{\frac{1}{\theta}} C_{j,t}^{1-\frac{1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}}$$

$$\int P_{j,t} C_{j,t} dj + Q_t B_t \leq B_{t-1} + W_t N_t + T_t$$

From the households problem, we obtain optimal conditions for

- Demand ($C_{j,t}$)

$$C_{j,t} = \left(\frac{P_{j,t}}{P_t} \right)^{-\theta} C_t \epsilon_{j,t}$$

  where the resulting aggregate price index

$$P_t \equiv \left[\int \epsilon_{j,t} P_{j,t}^{1-\theta} dj \right]^{\frac{1}{1-\theta}}$$

  implies $\int P_{j,t} C_{j,t} dj = P_t C_t$

- Labor supply (N_t)

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t}$$

$$\Psi C_t^{\gamma} = \frac{W_t}{P_t}$$

$$w_t - p_t = \gamma c_t + \log \Psi$$

[4] for non-linear disutility, see Appendix (7.7.1)

- Intertemporal consumption

$$Q_t = \beta \mathbb{E}_t \left(\frac{U_{c,t+1}}{U_{c,t}} \frac{P_t}{P_{t+1}} \right)$$
$$c_t = \mathbb{E}_t c_{t+1} - \frac{1}{\gamma}[i_t - \mathbb{E}_t \pi_{t+1} - \rho]$$

The representative household chooses labor N_t to maximize utility[5]

$$\max_{N_t} \frac{C_t^{1-\gamma}}{1-\gamma} + \Psi(1 - N_t)$$

subject to budget constraint

$$C_t \leq \frac{W_t}{P_t} N_t + \frac{\Pi_t}{P_t}$$

where C_t is aggregate an consumption index, $\frac{W_t}{P_t}$ is the real wage, $\frac{\Pi_t}{P_t}$ is real profit income from all firms, Ψ is disutility of labor. Their first order condition is

$$C_t^\gamma = \frac{1}{\Psi} \frac{W_t}{P_t} \tag{2}$$

where

$$C_t = \left[\int_0^1 \epsilon_{j,t}^{\frac{1}{\theta}} C_{j,t}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \tag{3}$$

C_t represents an aggregate consumption index, $\theta > 1$ is the elasticity of substitution between goods, $C_{j,t}$ denotes the quantity of good j consumed by the household in period t. The idiosyncratic preference shock for good j is log normally distributed ($\varepsilon_{j,t} \equiv \log \epsilon_{j,t} \sim N(0, \sigma_\varepsilon^2)$. The exponent $\frac{1}{\theta}$ on $\epsilon_{j,t}$ is solely intended to simplify calculations. The household allocates consumption among j goods to maximize C_t for any given level of expenditures $\int_0^1 P_{j,t} C_{j,t} dj$, where $P_{j,t}$ is the price of intermediate good j (see appendix (7.3)).

From optimizing its consumption allocation, household's demand for good j is given by

$$C_{j,t} = \left(\frac{P_t}{P_{j,t}} \right)^\theta C_t \epsilon_{j,t} \tag{4}$$

The resulting aggregate price level is obtained by substituting (4) into (3):

$$P_t = \left(\int_0^1 \epsilon_{j,t} P_{j,t} dj \right)^{\frac{1}{1-\theta}}$$

In this model, households form demand schedules for each differentiated good and supply labor, all contingent on shocks to idiosyncratic demand and shocks to aggregate income/consumption to be realized. Let Z_t represent the household's beliefs about aggregate

[5]Specifying the utility function in this way will allow sentiments to affect the real wage, by γ, which is ... This will affect the firms' marginal cost and their optimal response to sentiments. In the previous setup, $\gamma = 1$.

income/consumption at the beginning of period t. Households form consumption *plans* using (4)

$$C_{j,t}(Z_t, \epsilon_{j,t}) = \left(\frac{P_t(Z_t)}{P_{j,t}(Z_t, \epsilon_{j,t})}\right)^{\theta} C_t(Z_t)\epsilon_{j,t} \tag{5}$$

and decide labor supply, using (96) to obtain an implicit function of labor supply as a function of sentiments, $N_t = N(Z_t)$, given a nominal wage W_t

$$P_t(Z_t) = \frac{W_t}{\Psi\left[\frac{W_t}{P_t(Z_t)}N_t + \frac{\Pi_t(Z_t)}{P_t(Z_t)}\right]^{\gamma}} \tag{6}$$

Note that $\Pi_t(Z_t) = P_t(Z_t)Z_t - W_t N_t$.

3.2 Intermediate goods firms

Sentiment driven equilibria requires a signal extraction problem with two shocks, to each of which the optimal response of the firm's price setting decision is different. The Dixit-Stiglitz structure of the model implies that the optimal price for intermediate goods firm j under perfect information does not depend on the idiosyncratic preference shock for good j. To circumvent this, assume that a firm's marginal cost is positively correlated with its demand.

The intermediate goods firms decide price $P_{j,t}$ without perfect knowledge of idiosyncratic demand or aggregate demand. Instead, they infer $\epsilon_{j,t}$ and $Y_{j,t}$ from a signal $s_{j,t}$ that may be interpreted as early orders, advance sales, or market research:

$$s_{j,t} = \lambda\varepsilon_{j,t} + (1-\lambda)y_t$$

where $\varepsilon_{j,t} \equiv \log \epsilon_{j,t} \sim N(0, \sigma_\varepsilon^2)$ and $y_t \equiv (\log Y_t) - \phi_0 \sim N(0, \sigma_y^2)$.

Given an aggregate price index (P_t), intermediate goods producers choose $P_{j,t}$ to maximize nominal profits

$$\max_{P_{j,t}} \mathbb{E}_t \left[P_{j,t}Y_{j,t} - W_t N_{j,t}\right]$$

subject to production function

$$Y_{j,t} = \epsilon_{j,t}^{\tau} N_{j,t}$$

and demand schedule (imposing the market clearing condition, $C_t = Y_t$ and $C_{j,t} = Y_{j,t}$)

$$Y_{j,t} = \left(\frac{P_t}{P_{j,t}}\right)^{\theta} \epsilon_{j,t} Y_t$$

Substituting $N_{j,t}$ using firm j's production function and $Y_{j,t}$ from its demand schedule, the firms' problem is

$$\max_{P_{j,t}} \mathbb{E}_t \left[P_{j,t}^{1-\theta} P_t^{\theta} \epsilon_{j,t} Y_t - W_t P_t^{\theta} P_{j,t}^{-\theta} \epsilon_{j,t}^{1-\tau} Y_t | s_{j,t}\right] \tag{7}$$

The first order condition is given by:

$$(1-\theta)P_{j,t}^{-\theta}P_t^{\theta}\mathbb{E}_t(\epsilon_{j,t}Y_t|s_{j,t}) + \theta P_t^{\theta}P_{j,t}^{-\theta-1}\mathbb{E}_t(W_t\epsilon_{j,t}^{1-\tau}Y_t|s_{j,t}) = 0$$

As nominal variables are indeterminate in this model, the nominal aggregate consumption price index (P_t) can be normalized to 1. Rearranging terms:

$$P_{j,t} = \left(\frac{\theta}{\theta-1}\right)\frac{\mathbb{E}_t[W_t\epsilon_{j,t}^{1-\tau}Y_t|s_{j,t}]}{\mathbb{E}_t[\epsilon_{j,t}Y_t|s_{j,t}]}$$

Replacing W_t with the household's labor supply decision, firm j's optimal price is

$$P_{j,t} = \left(\frac{\theta}{\theta-1}\right)\Psi\frac{\mathbb{E}[\epsilon_{j,t}^{1-\tau}Y_t^{\gamma+1}|s_{j,t}]}{\mathbb{E}[\epsilon_{j,t}Y_t|s_{j,t}]}$$

3.3 Timing

The timing of this model is as follows: Let Z_t denote aggregate demand and $\epsilon_{j,t}$ represent idiosyncratic preference for good j:

1. Households form labor supply schedule ($N_t(Z_t)$) and demand schedules for each good j, ($C_{j,t}(Z_t, \epsilon_{j,t})$), contingent on shocks to be realized
2. Z_t, $\epsilon_{j,t}$ realized
3. Firms receive a private signal of aggregate demand and idiosyncratic preference for their good ($s_{j,t} = \lambda \log \epsilon_{j,t} + (1-\lambda)\log Z_t$)
4. Firms can not write contingent schedules for their labor demand, otherwise this would remove the possibility of sentiment-driven fluctuations. Instead, firms must commit to a price, based on an imperfect private signal. They set price $P_{j,t}(s_{j,t})$.
5. Goods market opens. Z_t, $\epsilon_{j,t}$ observed by everyone. Firms meet supply at posted price $Y_{j,t}(P_{j,t})$, so that goods market clears ($C_{j,t} = Y_{j,t}$, $C_t = Y_t$), and $W_t = \Psi Z_t^{\gamma}$.[6]

3.4 Equilibrium

In equilibrium, aggregate price index, intermediate goods price, and the private signal are given by:

$$P_t = \left[\int \epsilon_{j,t}P_{j,t}^{1-\theta}dj\right]^{\frac{1}{1-\theta}} \tag{8}$$

$$P_{j,t} = \frac{\theta}{\theta-1}\frac{\mathbb{E}_t[W_t\epsilon_{j,t}^{1-\tau}Y_t|s_{j,t}]}{\mathbb{E}_t[\epsilon_{j,t}Y_t|s_{j,t}]} \tag{9}$$

$$s_{j,t} = \lambda\log\epsilon_{j,t} + (1-\lambda)\log Y_t \tag{10}$$

(11)

[6] Thus, wages are realized at the end of the period.

Note that the firm's price setting decision already incorporates the household's optimal labor supply decision, $\frac{W_t}{P_t} = \Psi Y_t^{\gamma}$. In the sentiment driven equilibrium, one additional condition applies: that beliefs about aggregate demand are correct in equilibrium.

$$Z_t = Y_t \tag{12}$$

After the realization of Z_t, and after goods markets clear, market clearing quantities for each good, aggregate output, aggregate labor, nominal wage, and aggregate profits are given by:

$$Y_{j,t} = \left(\frac{P_t}{P_{j,t}}\right)^{\theta} \epsilon_{j,t} Y_t \tag{13}$$

$$Y_t = \left[\int \epsilon_{j,t}^{\frac{1}{\theta}} Y_{j,t}^{1-\frac{1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} \tag{14}$$

$$N_t = \int_0^1 N_{j,t} dj = \int_0^1 Y_{j,t} \epsilon_{j,t}^{-\tau} dj \tag{15}$$

$$\frac{W_t}{P_t} = \Psi Y_t^{\gamma} \tag{16}$$

$$\Pi_t = P_t Y_t - W_t N_t = Y_t - W_t N_t \tag{17}$$

The first equality follows from the household's demand equation and indicates that in equilibrium, the market clearing quantity of good j is determined by aggregate price index, price of good j, and realized aggregate output. The second follows from optimal aggregate consumption by households in conjunction with market clearing, the third from the firm's production function, and the fourth from the household's optimal labor supply condition. Finally, in the fifth equality, aggregate profits equal aggregate revenue minus aggregate production costs.

Definition 1. A rational expectations equilibrium is a sequence of allocations $\{C(Z_t), Y(Z_t), C_j(Z_t, \epsilon_{j,t}), Y_j(Z_t, \epsilon_{j,t}), N(Z_t), N_j(Z_t, \epsilon_{j,t}), \Pi(Z_t)\}$, prices $\{P_t = 1, P_j(Z_t, \epsilon_{j,t}), W_t = W(Z_t)\}$, and a distribution of Z_t, $\mathbf{F}(Z_t)$ such that for each realization of Z_t, (i) equations (99) and (100) maximize household utility given the equilibrium prices $P_t = 1, P_{j,t} = P_j(Z_t, \epsilon_{j,t})$, and $W_t = W(Z_t)$ (ii) equation (9) maximizes intermediate goods firm's *expected* profits for all j given the equilibrium prices $P_t = 1, W_t = W(Z_t)$, and the signal (104) (iii) all markets clear: $C_{j,t} = Y_{j,t}, N(Z_t) = \int N_{j,t} dj$, and (iv) expectations are rational such that the household's beliefs about W_t and Π_t are consistent with its belief about aggregate demand Z_t (according to its optimal labor supply condition) and $Y_t = Z_t$: actual aggregate output follows a distribution consistent with $\mathbf{F}$.

There exist two rational expectations equilibria: (1) A fundamental equilibrium with a degenerate distribution of sentiments, where aggregate output and prices are all constant and where sentiments play no role in determining the level of aggregate output (2) A stochastic equilibrium where sentiments matter and the volatility of beliefs about aggregate demand is endogenously determined and equal to the variance of aggregate output.

3.4.1 Fundamental equilibrium

Under perfect information, there is a unique rational expectations equilibrium in which the price of good j, aggregate price level, and aggregate demand are constant. aggregate output is constant and known. Then, the private signal that firms receive reveals their idiosyncratic demand shocks. Using the equilibrium conditions in (9), (102), (107), and (106), $Y_t, P_t, Y_{j,t}$ and $P_{j,t}$ in the fundamental equilibrium are as follows:

With perfect information, the price of good j (9):

$$P_{j,t} = \frac{\theta}{\theta - 1} \frac{W_t \epsilon_{j,t}^{1-\tau} Y_t}{\epsilon_{j,t} Y_t}$$

Replacing W_t with (2),

$$P_{j,t} = \frac{\theta}{\theta - 1} \Psi P_t Y_t^{\gamma} \epsilon_{j,t}^{-\tau}$$

Without loss of generality, normalizing $\frac{\theta}{\theta-1}\Psi$ to 1,

$$P_{j,t} = P_t Y_t^{\gamma} \epsilon_{j,t}^{-\tau} \tag{18}$$

Substituting (18) into (8), the aggregate price index in the fundamental equilibrium is indeterminate:

$$\begin{aligned} P_t &= \left[\int \epsilon_{j,t} [P_t Y_t^{\gamma} \epsilon_{j,t}^{-\tau}]^{1-\theta} dj \right]^{\frac{1}{1-\theta}} \\ &= \left[\int \epsilon_{j,t}^{1-\tau(1-\theta)} dj \right]^{\frac{1}{1-\theta}} P_t Y_t^{\gamma} \end{aligned}$$

Without loss of generality, normalize P_t to 1.
Next, the normalization of $P_t = 1$ can be used to find Y_t,

$$Y_t = \left[\int \epsilon_{j,t}^{1-\tau(1-\theta)} dj \right]^{\frac{1}{\gamma(\theta-1)}} \tag{19}$$

Taking the log of this expression (let $y_t \equiv (\log Y_t) - \phi_0$),

$$y_t + \phi_0 = \frac{1}{\gamma(\theta - 1)} \log \mathbb{E}_t \left[\epsilon_{j,t}^{1-\tau(1-\theta)} \right]$$

As $\varepsilon_{j,t} \equiv \log \epsilon_{j,t} \sim N(0, \sigma_{\varepsilon}^2)$, by the properties of a moment generating function for a normally distributed random variable,

$$\begin{aligned} y_t + \phi_0 &= \frac{1}{\theta - 1} \frac{1}{2} \mathrm{Var}_t([1 - \tau(1-\theta)]\varepsilon_{j,t}) && (20) \\ &= \frac{1}{\gamma(\theta - 1)} \frac{[1 - \tau(1-\theta)]^2}{2} \sigma_{\varepsilon}^2 && (21) \end{aligned}$$

Equating coefficients implies $y_t = 0$ and

$$\phi_0^* = \frac{1}{2(\theta - 1)} \frac{(1 + \tau[\theta - 1])^2}{\gamma} \sigma_{\varepsilon}^2$$

Note that output in the fundamental equilibrium when firms choose quantity (112), ($\gamma = 1, \tau = 0$) is equivalent to its counterpart when firms choose prices. In Appendix (), the fundamental equilibrium is derived when firms choose quantity, households face a general utility function, and firms face a general production function, as in this price-setting section.:

$$\phi_0^* = \frac{1}{2(\theta-1)} \frac{(1+\tau[\theta-1])^2}{\gamma} \sigma_\varepsilon^2 \tag{22}$$

Finally, an expression for $Y_{j,t}$ can be found by using the demand curve (13), and substituting $P_{j,t}$ with (18)

$$\begin{aligned} Y_{j,t} &= \left(\frac{P_t}{P_{j,t}}\right)^\theta \epsilon_{j,t} Y_t \\ &= [Y_t^\gamma \epsilon_{j,t}^{-\tau}]^{-\theta} \epsilon_{j,t} Y_t \\ &= \epsilon_{j,t}^{1+\tau\theta} Y_t^{1-\sigma\theta} \end{aligned}$$

Replacing Y_t with (19),

$$Y_{j,t} = \epsilon_{j,t}^{1+\tau\theta} \left[\int \epsilon_{j,t}^{1-\tau(1-\theta)} dj\right]^{\frac{1-\gamma\theta}{\gamma(\theta-1)}}$$

Report $y_{j,t}$

3.4.2 Sentiment-driven equilibrium

When firms face information frictions, there exists a sentiment driven equilibrium, in addition to the fundamental equilibrium. The sentiment driven equilibrium is a rational expectations equilibrium where aggregate output is not constant but equal to a sentiment (Z_t). Let $\hat{z}_t$ and $\hat{y}_t$ denote Z_t and Y_t in log deviation from the steady state of this equilibrium, respectively.[7] $\hat{z}_t \sim N(0, \sigma_z^2)$, where σ_z^2 is a constant to be determined below.

Intuition: Consider the case of a positive sentiment shock in the flexible wage and flexible price model. A self-fulfilling equilibrium is possible when σ_z^2 is sufficiently low such that firms attribute just enough of z_t to $\epsilon_{j,t}$ and so that the increase in sentiment leads firms to lower $p_{j,t}$. When goods markets open, the quantity of firm j's product, demanded at price $p_{j,t}$, ($y_{j,t}(p_{j,t})$), is higher than that under perfect information. Thus, there is a σ_z^2 such that aggregate supply across firms exactly fulfills the positive sentiment formed by households.

Proposition 1. There exists a sentiment-driven rational expectations equilibrium with stochastic aggregate output $\log Y_t = y_t + \phi_0 = z_t + \phi_0$, that has mean $\phi_0 =$ and $\sigma_z^2 = \frac{\lambda}{1-\lambda}\frac{\tau}{\gamma}\sigma_\varepsilon^2$.

Proof. Key steps:

1. From the firms' optimal price setting ($p_{j,t}$), take logs and simplify expressions
2. Guess $p_{j,t}$ and from this, deduce p_t
3. Either use guess for $p_{j,t}$ to solve for P_t using the aggregate price index, or use guess for p_t to solve for $P_{j,t}$ using what was obtained in step (1)

[7] See appendix (7.4) for a calculation of the steady state in this equilibrium.

4. Equate coefficients and determine the steady state and sentiment volatility NOTE: Prefer to guess p_t and substitute this

Equation (9) gives firm j's optimal price conditional on its signal. As it is derived using equations (16) and (13), it already incorporates market clearing for labor and consumption.

$$
\begin{aligned}
P_{j,t} &= \frac{\theta}{\theta-1}\frac{\mathbb{E}_t[W_t\epsilon_{j,t}^{1-\tau}Y_t|s_{j,t}]}{\mathbb{E}_t[\epsilon_{j,t}Y_t|s_{j,t}]} \\
&= \frac{\theta}{\theta-1}\Psi\frac{\mathbb{E}_t[P_t\epsilon_{j,t}^{1-\tau}Z_t^{\gamma+1}|s_{j,t}]}{\mathbb{E}_t[\epsilon_{j,t}Z_t|s_{j,t}]}
\end{aligned}
$$

where the second equality results from substituting W_t with the household's optimal labor supply (16). Taking logs,

$$
p_{j,t} = \log\left(\frac{\theta}{\theta-1}\Psi\right) + \log\mathbb{E}_t[P_t\epsilon_{j,t}^{1-\tau}Z_t^{\gamma+1}|s_{j,t}] - \log\mathbb{E}_t[\epsilon_{j,t}Z_t|s_{j,t}]
$$

Guess a solution of the form $p_{j,t} = D + Bs_{j,t}$. According to this guess, $p_t = A + B(1-\lambda)z_t$ where A incorporates $\mathbb{E}(\epsilon_{j,t})$, which affects the steady state. Substituting our guess for p_t,

$$
p_{j,t} = \log\left(\frac{\theta}{\theta-1}\Psi\right) + \log\mathbb{E}_t[\exp(p_t + (1-\tau)\varepsilon_{j,t} + (\gamma+1)(z_t+\phi_0))|s_{j,t}] - \log\mathbb{E}_t[\exp(\varepsilon_{j,t} + z_t + \phi_0)|s_{j,t}] \tag{23}
$$

$$
= \log\left(\frac{\theta}{\theta-1}\Psi\right) + \gamma\phi_0 + A + \log\mathbb{E}[\exp(B(1-\lambda)+\gamma+1)z_t + (1-\tau)\varepsilon_{j,t}|s_{j,t}] - \log\mathbb{E}_t[\exp(\varepsilon_{j,t}+z_t)] \tag{24}
$$

$$
= \log\left(\frac{\theta}{\theta-1}\Psi\right) + \gamma\phi_0 + A + \frac{\Omega_1-\Omega_2}{2} + (\mu_1-\mu_2)s_{j,t} \tag{25}
$$

$$
= \varphi_0 + \bar{\mu}s_{j,t} \tag{26}
$$

where

$$
\varphi_0 \equiv \log\left(\frac{\theta}{\theta-1}\Psi\right) + \gamma\phi_0 + A + \frac{\Omega_1-\Omega_2}{2} \tag{27}
$$

$$
\bar{\mu} \equiv \mu_1 - \mu_2 \tag{28}
$$

$$
\mu_1 \equiv \mathbb{E}_t[B(1-\lambda)+\gamma+1)z_t + (1-\tau)\varepsilon_{j,t}|s_{j,t}] \tag{29}
$$

$$
\Omega_1 \equiv \frac{1}{2}\mathrm{Var}_t[B(1-\lambda)+\gamma+1)z_t + (1-\tau)\varepsilon_{j,t}|s_{j,t}] \tag{30}
$$

$$
\mu_2 \equiv \mathbb{E}_t[\varepsilon_{j,t}+z_t|s_{j,t}] \tag{31}
$$

$$
\Omega_2 \equiv \frac{1}{2}\mathrm{Var}[\varepsilon_{j,t}+z_t|s_{j,t}] \tag{32}
$$

Variables in lowercase denote the log of their counterparts, with the exception of $z_t = \log Z_t - \phi_0$. Note that the firm's price is a constant projection of $s_{j,t}$. Hence, in a sentiment-driven equilibrium, all firms set prices in the same proportion to their signal.

Taking the log of the aggregate price index (8) and substituting for $p_{j,t}$ with (26),

$$\begin{aligned}(1-\theta)p_t &= \log \mathbb{E}_t[P_{j,t}^{1-\theta}\epsilon_{j,t}] \\ &= \log \mathbb{E}_t[\exp([1-\theta]p_{j,t} + \varepsilon_{j,t})] \\ &= (1-\theta)\varphi_0 + (1-\theta)\bar{\mu}(1-\lambda)z_t + \log \mathbb{E}_t[e^{([1-\theta]\bar{\mu}\lambda+1)\varepsilon_{j,t}}] \\ A + Bz_t &= \varphi_0 + \bar{\mu}(1-\lambda)z_t + \frac{[(1-\theta)\bar{\mu}\lambda+1]^2}{2(1-\theta)}\sigma_\epsilon^2\end{aligned}$$

Equating coefficients on z_t,

$$B = \bar{\mu}(1-\lambda) \tag{33}$$

Evaluating (29) and (31), we have

$$B = \frac{(\gamma + B)(1-\lambda)\sigma_z^2 - \tau\lambda(1-\lambda)\sigma_\epsilon^2}{\lambda^2\sigma_\epsilon^2 + (1-\lambda)^2\sigma_z^2}(1-\lambda)$$

which implies[8]

$$\sigma_z^2 = \frac{\lambda}{1-\lambda}\frac{\tau + B\frac{\lambda}{1-\lambda}}{\gamma}\sigma_\epsilon^2$$

Thus, the volatility of actual aggregate output and beliefs about aggregate demand are determined by the parameters of the model. If $\lambda \in (0,1)$, $\tau > 0$, and $\sigma_\varepsilon^2 > 0$, then there exists a sentiment driven rational expectations equilibrium with $\hat{y}_t = \hat{z}_t$ where

$$\sigma_y^2 = \sigma_z^2 \tag{34}$$

□

Factors affecting sentiment volatility:

- The volatility of the sentiment shock must be commensurate with the degree of complementarity/substitutability in actions across firms (τ, γ), [9] information content of the private signal (λ), and the volatility of idiosyncratic demand (σ_ε^2), all of which affect the firm's response to a sentiment shock.

- Note that if $\tau = 0, \lambda = 0$ or $\sigma_\varepsilon^2 = 0$, then the private signal conveys only aggregate demand or price depends only on aggregate demand. The result is also that the unique equilibrium is the fundamental equilibrium, due to substitutability of firms' outputs.

- Sentiment volatility is decreasing in $1-\lambda$; as the private signal becomes more informative about aggregate demand ($1-\lambda$ increases), we approach the certainty equilibrium of the previous section.

[8] The relationship between the price level and sentiments is indeterminate. In BWW, P_t is normalized to 1, and so $B = \frac{p_{j,t}}{z_t} = \frac{p_t}{z_t} = 0$.

[9] **why, details about how it affects MC**

- Sentiment volatility is increasing in $\sigma_\varepsilon^2 = 0$; a sentiment driven equilibrium needs sufficient coordination: all firms set the same price regardless of their individual signal, but depending on the (known) distribution of signals. The more volatile the idiosyncratic component of the signal, the more difficult it is to attain coordination. In this case, sentiment volatility must be commensurately larger.

The intuition for why the sentiment-driven equilibrium is a rational expectations equilibrium is as follows: Given the parameters of the model, σ_z^2 is determined such that for any aggregate demand sentiment, all firms misattribute enough of the sentiment component of their signal to an idiosyncratic preference shock such that price-setting decisions lead to aggregate output equaling the sentiment in equilibrium. The volatility of the sentiment process (σ_z^2) determines how much firms attribute their signal to $\hat{z}_t$. Firms increase their price in response to aggregate demand, and decrease their price in response to idiosyncratic demand. Through prices, firms' output decision are strategic substitutes. When firms actions are strategic substitutes, the optimal output of a firm is declining in σ_z^2 as this leads the firms to attribute more of the signal to an aggregate demand shock. Since firms' optimal price depends negatively on the idiosyncratic preference shock $\hat{\varepsilon}_{j,t}$ and positively on the level of aggregate demand, $\hat{z}_t$, if they are unable to distinguish between the two components in their signal, then there can be a coordinated over-production (under-production) in response to a positive (negative) aggregate sentiment shock, such that $\hat{y}_t$ equals $\hat{z}_t$ in equilibrium if σ_z^2 is as in (**??**). The rational expectations equilibrium pins down the variance of the sentiment distribution, although sentiments are extrinsic. The result is an additional rational expectations equilibrium that is characterized by aggregate fluctuations in output and employment despite the lack of fundamental aggregate shocks.
From equating the constant terms, we have

$$A = \varphi_0 + \frac{[(1-\theta)\bar{\mu}\lambda + 1]^2}{2(1-\theta)}\sigma_\epsilon^2$$

Applying (33) and (87),

$$\phi_0 = \frac{1}{\gamma}\left(\frac{[(1-\theta)\frac{\lambda}{1-\lambda}B + 1]^2}{2(\theta-1)}\sigma_\epsilon^2 - \log\left(\frac{\theta}{\theta-1}\Psi\right) - \frac{\Omega_1 - \Omega_2}{2}\right)$$

Note that A is the steady state for the price level, which is indeterminate, while ϕ_0 is the steady state for aggregate output. The conditional variances are constants, and functions of σ_ϵ^2, σ_z^2, and other parameters of the model:

$$\Omega_1 - \Omega_2 = [(\gamma + B)^2 + (2-\mu_1)(\gamma + B) - B]\sigma_z^2 + \left[\tau^2 + (\mu_1 - 2)\tau - B\frac{\lambda}{1-\lambda}\right]\sigma_\epsilon^2$$

4 Monetary Policy with Sticky Prices (Calvo)

Flexible wages and sticky prices

- HH's problem is the same

- However, in any given period, a fraction θ_p of firms can not adjust their price. Instead, $(1-\theta_p)$ of firms choose their optimal price taking into account the probability of not being able to adjust for $\frac{1}{\theta_p}$ periods.
- As we will see, it will be more difficult to achieve coordination when prices are sticky. As a result, sentiment driven fluctuations are less volatile. Due to the endogeneity of sentiment volatility, the result is that when the CB targets inflation strongly or prices are more flexible, this leads to higher volatility of output.
- Note that although sentiment shocks are *iid* (and thus price setting with sticky prices is equivalent to price setting under flexible prices), the Calvo parameter affects inflation through the proportion of firms who can reset prices.

Intuition: In a sentiment driven equilibrium with sticky prices and monetary policy, the self-fulfilling equilibrium has a different mechanism than in the case with flexible wages and prices.
The first section will consider the micro-foundations of the baseline model (decisions of households, firms, equilibrium condition). The quantity of output in the fundamental equilibrium is derived, followed by the mean level of output in the sentiment driven equilibrium. In addition, the mechanism behind a self-fulfilling equilibrium with sentiments (how sentiments lead to fluctuations in output) will be described.
The second section will consider the impact of monetary policy on the volatility of output in the sentiment-driven equilibrium.
"Production is split into two stages: intermediate and final goods. The final goods production technology is simply a constant elasticity (CES) bundler of intermediate goods– there are no factors (i.e. labor) used to produce final goods. Profit maximization in the final goods sector (which is competitive) yields a downward sloping demand curve for intermediate goods producers, which gives them some pricing power. It is in the intermediate goods sector that we will assume some nominal rigidity (i.e. price-stickiness), which is in turn capable of generating meaningful non-neutralities."

4.1 Households

same as benchmark case

4.2 Firms

4.2.1 Optimal Price Setting

- **Marginal cost** We derive the firms' marginal cost from the following minimization problem:

  $$\min_{N_{j,t}} W_t N_{j,t}$$

  subject to

  $$Y_{j,t} \leq \epsilon^{\tau}_{j,t} N_{j,t}$$

  The Lagrangian is

  $$L = W_t N_{j,t} - \Phi_t(\epsilon^{\tau}_{j,t} N_{j,t} - Y_{j,t})$$

Substituting for W_t using (HH's opt ls), nominal marginal cost is

$$\Phi_t = \Psi \epsilon_{j,t}^{-\tau} Z_t^{\gamma} P_t$$
$$\phi_t = \log(\Psi) - \tau\epsilon_{j,t} + \gamma z_t + p_t$$

- **NKPC** With Calvo price setting, the aggregate price index is as follows:

$$P_t^{1-\theta} = \int_{\times_t^c} P_{j,t}^{1-\theta}\epsilon_{j,t}dj + \int_{\times_t} P_{j,t}^{*(1-\theta)}\epsilon_{j,t}dj$$

where $\times_t^c$ denotes the set of firms who can not re-adjust prices in period t and $\times_t$ as the complement of this set. Let

$$P_{t-1}^{1-\theta} \equiv \frac{1}{\theta_p}\int_{\times_t^c} P_{j,t}^{1-\theta}\epsilon_{j,t}dj \tag{35}$$

$$P_t^{*(1-\theta)} \equiv \frac{1}{1-\theta_p}\int_{\times_t} P_{j,t}^{*(1-\theta)}\epsilon_{j,t}dj \tag{36}$$

Then

$$P_t^{1-\theta} = \theta_p P_{t-1}^{1-\theta} + (1-\theta_p)P_t^{*(1-\theta)} \tag{37}$$

$$\Pi_t^{1-\theta} = \theta_p + (1-\theta_p)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\theta} \tag{38}$$

A first order approximation to (38) around a zero inflation steady state yields

$$\pi_t = (1-\theta_p)(p_t^* - p_{t-1}) \tag{39}$$

The firm's profit-maximizing price is (p 47, Gali)

$$p_{j,t}^* - p_{t-1} = (1-\beta\theta_p)\mathbb{E}_t[\gamma z_t - \tau\varepsilon_{j,t}|s_{j,t}] + \mathbb{E}_t[\pi_t|s_{j,t}]$$

Substituting π_t with (39),

$$p_{j,t}^* - p_{t-1} = (1-\beta\theta_p)\mathbb{E}_t[\gamma z_t - \tau\varepsilon_{j,t}|s_{j,t}] + \mathbb{E}_t[\pi_t|s_{j,t}] \tag{40}$$

To solve, key steps are

1. Conjecture $p_t^* = \tilde{D} + \mu(1-\lambda)z_t$
2. Use conjecture and (40) to find $p_{j,t}^*$

$$\begin{aligned} p_{j,t}^* &= (1-\beta\theta_p)\mathbb{E}[\gamma z_t - \tau\varepsilon_{j,t}|s_{j,t}] + (1-\theta_p)\mathbb{E}_t[\tilde{D} + \mu(1-\lambda)z_t|s_{j,t}] + \theta_p p_{t-1} \\ &= (1-\theta_p)\tilde{D} + \theta_p p_{t-1} + \mathbb{E}_t([(1-\beta\theta_p)\gamma + (1-\theta_p)\mu(1-\lambda)]z_t - (1-\beta\theta_p)\tau\varepsilon_{j,t}|s_{j,t}) \end{aligned}$$

Let $p_{j,t}^* = D + \mu s_{j,t}$ where

$$D \equiv (1-\theta_p)\tilde{D} + \theta_p p_{t-1}$$
$$\mu \equiv \frac{\text{cov}([(1-\beta\theta_p)\gamma + (1-\theta_p)\mu(1-\lambda)]z_t - (1-\beta\theta_p)\tau\varepsilon_{j,t}, s_{j,t})}{\text{var}(s_{j,t})}$$

3. Substitute $p^*_{j,t}$ into (36) and equate coefficients to find the steady state for $p^*_{j,t}$ and p^*_t, as well as their responses to z_t.
Taking the log of (36) and defining $\mathbb{E}_{\times_t}$ as $\frac{1}{1-\theta_p}\int_{\times_t}$,

$$(1-\theta)p^*_t = \ln \mathbb{E}_{\times_t} e^{(1-\theta_p)p^*_{j,t}+\varepsilon_{j,t}}$$
$$p^*_t = D + \mu(1-\lambda)z_t + \frac{[(1-\theta)\mu\lambda+1]^2}{2(1-\theta)}\sigma^2_\epsilon$$

Equating coefficients,

$$\tilde{D} = p_{t-1} + \frac{1}{\theta_p}\frac{[(1-\theta)\mu\lambda+1]^2}{2(1-\theta)}\sigma^2_\epsilon$$
$$D = p_{t-1} + \frac{1-\theta_p}{\theta_p}\frac{[(1-\theta)\mu\lambda+1]^2}{2(1-\theta)}\sigma^2_\epsilon$$
$$\mu = (1-\beta\theta_p)\frac{\gamma(1-\lambda)\sigma^2_z - \tau\lambda\sigma^2_\epsilon}{\lambda^2\sigma^2_\epsilon + \theta_p(1-\lambda)^2\sigma^2_z}$$

Note that μ is close to $\mathbb{E}_t[\gamma z_t - \tau\varepsilon_{j,t}|s_{j,t}]$ if $\theta_p \to 1$. The more flexible prices are ($\theta_p \to 0$), the larger is μ, and the more pass through of z_t to $p^*_{j,t}$ and thus to p^*_t. When prices are sticky, coordination is more difficult to achieve. The θ_p in the denominator is from the effect of z_t on p^*_t. The more p^*_t is composed of the non-fundamental component, the more μ will weight $s_{j,t}$ to pass through more of z_t. As a result, $p^*_{j,t}$ contains more of the non-fundamental component. The implied processes are

$$p^*_{j,t} = p_{t-1} + \frac{1-\theta_p}{\theta_p}\frac{[(1-\theta)\mu\lambda+1]^2}{2(1-\theta)}\sigma^2_\epsilon + (1-\beta\theta_p)\frac{\gamma(1-\lambda)\sigma^2_z - \tau\lambda\sigma^2_\epsilon}{\lambda^2\sigma^2_\epsilon + \theta_p(1-\lambda)^2\sigma^2_z}s_{j,t} \tag{41}$$

$$p^*_t = p_{t-1} + \frac{1}{\theta_p}\frac{[(1-\theta)\mu\lambda+1]^2}{2(1-\theta)}\sigma^2_\epsilon + (1-\beta\theta_p)\frac{\gamma(1-\lambda)\sigma^2_z - \tau\lambda\sigma^2_\epsilon}{\lambda^2\sigma^2_\epsilon + \theta_p(1-\lambda)^2\sigma^2_z}(1-\lambda)z_t \tag{42}$$

Substituting for p^*_t in (39) with (42), we get a form of the NKPC, which results from the price setting behavior of firms with imperfect information.

$$\pi_t = \frac{1-\theta_p}{\theta_p}\frac{[(1-\theta)\mu\lambda+1]^2}{2(1-\theta)}\sigma^2_\epsilon + (1-\theta_p)(1-\beta\theta_p)\frac{\gamma(1-\lambda)\sigma^2_z - \tau\lambda\sigma^2_\epsilon}{\lambda^2\sigma^2_\epsilon + \theta_p(1-\lambda)^2\sigma^2_z}(1-\lambda)z_t \tag{43}$$

Note that the degree of pass through of z_t to π_t is increasing in the degree of price flexibility ($\theta_p \downarrow$).

4.3 Central bank

The central bank sets the nominal interest rate as a function of price inflation and output

$$i_t = \rho + \phi_\pi\pi_t + \phi_y y_t$$

4.4 Equilibrium

The only difference from the benchmark (flexible price) equilibrium is in the optimal price setting equation In equilibrium, aggregate price index, intermediate goods price, and the private signal are given by:

$$P_t = \left[\int \epsilon_{j,t} P_{j,t}^{1-\theta} dj \right]^{\frac{1}{1-\theta}} \tag{44}$$

$$0 = \sum_{k=0}^{\infty} \theta_p^k \mathbb{E}_t [Q_{t,t+k} Y_{t+k|t} (P_{j,t}^* - M\psi_{t+k|t})] \tag{45}$$

$$s_{j,t} = \lambda \log \epsilon_{j,t} + (1-\lambda) \log Y_t \tag{46}$$

With *iid* sentiments, (45) simplies to

$$P_{j,t}^* = \frac{\theta}{\theta - 1} \frac{\mathbb{E}_t[W_t \epsilon_{j,t}^{1-\tau} Y_t | s_{j,t}]}{\mathbb{E}_t[\epsilon_{j,t} Y_t | s_{j,t}]}$$

In the sentiment driven equilibrium, one additional condition applies: that beliefs about aggregate demand are correct in equilibrium.

$$Z_t = Y_t \tag{47}$$

After the realization of Z_t, and after goods markets clear, market clearing quantities for each good, aggregate output, aggregate labor, nominal wage, and aggregate profits are given by:

$$Y_{j,t} = \left(\frac{P_t}{P_{j,t}} \right)^{\theta} \epsilon_{j,t} Y_t \tag{48}$$

$$Y_t = \left[\int \epsilon_{j,t}^{\frac{1}{\theta}} Y_{j,t}^{1-\frac{1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \tag{49}$$

$$N_t = \int_0^1 N_{j,t} dj = \int_0^1 Y_{j,t} \epsilon_{j,t}^{-\tau} dj \tag{50}$$

$$\frac{W_t}{P_t} = \Psi Y_t^{\gamma} \tag{51}$$

$$\Pi_t = P_t Y_t - W_t N_t = Y_t - W_t N_t \tag{52}$$

The first equality follows from the household's demand equation and indicates that in equilibrium, the market clearing quantity of good j is determined by aggregate price index, price of good j, and realized aggregate output. The second follows from optimal aggregate consumption by households in conjunction with market clearing, the third from the firm's production function, and the fourth from the household's optimal labor supply condition. Finally, in the fifth equality, aggregate profits equal aggregate revenue minus aggregate production costs.

4.5 Effect of an *iid* shock to sentiments

Two equations that describe how inflation is related to sentiments (1) the NKPC (43) and (2) Euler equation and Taylor rule:

$$\pi_t = -\frac{\gamma + \phi_y}{\phi_\pi} z_t \tag{53}$$

In a sentiment driven equilibrium, the σ_z^2 that satisfies both relationships is

$$\sigma_z^2 = \frac{\lambda}{1-\lambda} \frac{\tau - \frac{\lambda}{1-\lambda} \frac{1}{(1-\beta\theta_p)(1-\theta_p)} \frac{\gamma+\phi_y}{\phi_\pi}}{\gamma + \frac{\theta_p}{(1-\beta\theta_p)(1-\theta_p)} \frac{\gamma+\phi_y}{\phi_\pi}} \sigma_\epsilon^2 \tag{54}$$

Note that as price flexibility facilitates the pass through of z_t, sentiment volatility is increasing in the degree to which firms are able to adjust prices. Need better explanation for ϕ_π.

4.6 Discussion

The CB can suppress these non-fundamental fluctuations with sufficiently lax monetary policy: If $\phi_\pi < \frac{\lambda}{1-\lambda} \frac{1}{\theta_p \lambda_p} \frac{\gamma+\phi_y}{\tau}$. Intuition or this: if ϕ_π is weak enough, firms pay more attention to z_t, are more sensitive to it in their signal and are less likely to pass it through in their actions. When monetary policy is too aggressive, reacting too strongly to inflation or output, this dis-incentivizes firms to pay attention to the z_t component of their signal, since the penalty (higher real wage ex-post) for not doing so is lower. When CB relaxes their approach to inflation, they force the economy to build its own defenses (stabilization mechanisms).

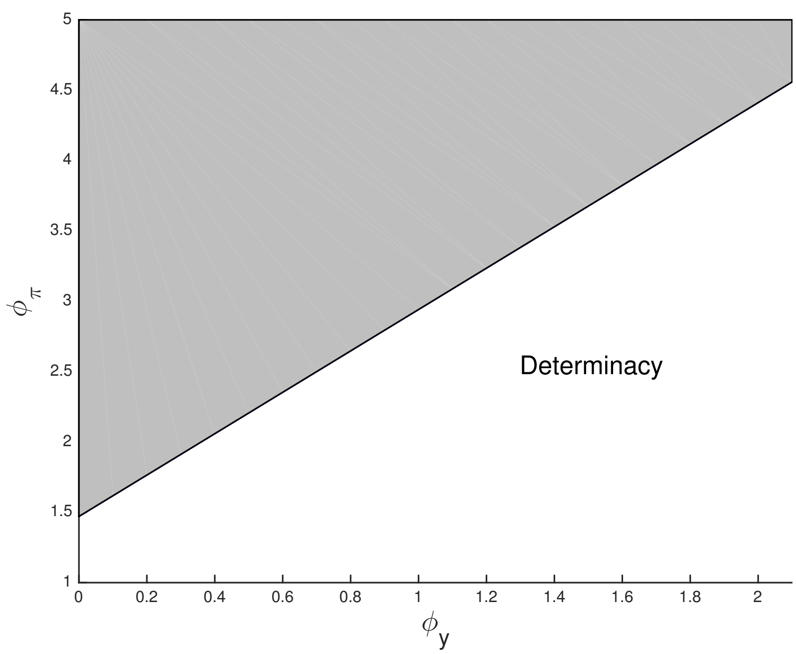


Figure 1: $\theta_p = \frac{2}{3}$, $\beta = 0.99$, $\gamma = 1$, $\lambda = 0.2$, $\tau = 1$

5 Productivity shock (A_t)

5.1 Setup

The firm's problem is to

$$\max_{Y_{j,t}} \mathbb{E}_t[P_{j,t}Y_{j,t} - W_t N_{j,t}|s_{j,t}]$$

subject to

$$\begin{aligned}
Y_{j,t} &= A_{j,t}^{\tau} N_{j,t} \\
Y_{j,t} &= \left(\frac{P_t}{P_{j,t}}\right)^{\theta} \epsilon_{j,t} Y_t \\
\frac{W_t}{P_t} &= Z_t^{\gamma} \\
A_{j,t} &= A_t^{\eta} \epsilon_{j,t}^{1-\eta} \\
s_{j,t} &= \lambda \varepsilon_{j,t} + (1-\lambda) z_t
\end{aligned}$$

$$Y_{j,t}^{\frac{1}{\theta}} = \frac{\theta}{\theta-1}\frac{1}{\Psi}\frac{\mathbb{E}_t[\epsilon_{j,t}^{\frac{1}{\theta}} Z_t^{\frac{1}{\theta}}|s_{j,t}]}{\mathbb{E}_t[Z_t^{\gamma} A_t^{-\tau\eta} \epsilon_{j,t}^{-\tau(1-\eta)}|s_{j,t}]} \tag{55}$$

When persistence is introduced, the aggregate technology shock follows an AR(1) process,

$$A_t = A_{t-1}^{\rho} \epsilon_t^a$$

In logs,

$$\begin{aligned}
a_{j,t} &= \eta a_t + (1-\eta)\varepsilon_{j,t} \\
a_t &= \rho a_{t-1} + \varepsilon_t^a
\end{aligned}$$

where $\varepsilon_t^a \sim N(0, \sigma_a^2)$, which implies $\text{Var}(a_t) = \frac{\sigma_a^2}{1-\rho^2}$
The solution method is as follows:

1. (a) Conjecture $Z_t = A_t^{\psi_{za}} \nu_y$ to solve for σ_a^2. Obtain $\sigma_z^2 = \psi_{za}^2 \sigma_a^2$, or (b) conjecture $A_t = Z_t^{\frac{1}{\psi_{za}}} \nu_y^{-\frac{1}{\psi_{za}}}$ and solve for σ_z^2.
2. Get an expression for $Y_{j,t}$ from the firms' first order condition (55).
3. Substitute $Y_{j,t}$ in to aggregate goods index

$$\begin{aligned}
\mu_1 &\equiv \mathbb{E}[\frac{1}{\theta}\varepsilon_{j,t} + \psi_{za}\frac{1}{\theta} a_t | s_{j,t}] \\
\mu_2 &\equiv \mathbb{E}[(\psi_{za}\gamma - \tau\eta) a_t - \tau(1-\eta)\varepsilon_{j,t}|s_{j,t}]
\end{aligned}$$

4. Equate coefficients With $\nu_y = \phi_0$,

$$\phi_0 = \frac{1}{\gamma}\left[\log\left(\frac{\theta}{\theta-1}\frac{1}{\Psi}\right) + \frac{\Omega_1 - \Omega_2}{2} + \frac{1}{2\theta(\theta-1)}\left(\frac{\theta-1}{\theta}\frac{\lambda}{1-\lambda} + \frac{1}{\theta}\right)^2 \sigma_\epsilon^2\right]$$

$$\theta(\mu_1 - \mu_2) = \frac{1}{1-\lambda}$$

which implies

$$\frac{[1+\tau\theta(1-\eta)]\lambda\sigma_\epsilon^2 + [\psi_{za} - \theta(\psi_{za}\gamma - \tau\eta)]\psi_{za}(1-\lambda)\sigma_a^2}{\lambda^2\sigma_\epsilon^2 + \psi_{za}^2(1-\lambda)^2\sigma_a^2} = \frac{1}{1-\lambda} \tag{56}$$

The effect of the aggregate productivity shock is evident in (56): it reduces the degree of substitutability in firm production with respect to aggregate output, as aggregate technology shocks (A_t) are positively correlated with firm level productivity ($A_{j,t}$). $-\tau\eta\psi_{za}$ captures the degree to which A_t mitigates the firms' negative response to aggregate output (Z_t).

From (56), obtain[10]

$$\sigma_a^2 = \frac{\lambda}{1-\lambda}\frac{\tau\theta(1-\eta)+1-\frac{\lambda}{1-\lambda}}{\theta(\gamma-\frac{\tau\eta}{\psi_{za}})}\frac{1}{\psi_{za}^2}\sigma_\epsilon^2$$

which implies

$$\sigma_z^2 = \frac{\lambda}{1-\lambda}\frac{\tau\theta(1-\eta)+1-\frac{\lambda}{1-\lambda}}{\theta(\gamma-\frac{\tau\eta}{\psi_{za}})}\sigma_\epsilon^2 \tag{57}$$

(56) also gives a quadratic equation in ψ_{za}

$$\gamma\psi_{za}^2 - \tau\eta\psi_{za} - \frac{\lambda}{1-\lambda}\left[\tau(1-\eta) + \frac{1}{\theta}(1-\frac{\lambda}{1-\lambda})\right]\frac{\sigma_\epsilon^2}{\sigma_a^2} = 0 \tag{58}$$

$$\begin{aligned}\psi_{za} &= \frac{\tau\eta}{2\gamma} \pm \sqrt{\left(\frac{\tau\eta}{2\gamma}\right)^2 + \frac{\lambda}{1-\lambda}\frac{\tau(1-\eta)+1-\frac{\lambda}{1-\lambda}}{\gamma\theta}\frac{\sigma_\epsilon^2}{\sigma_a^2}} \\ &= \frac{\tau\eta}{2\gamma} \pm \sqrt{\left(\frac{\tau\eta}{2\gamma}\right)^2 + \frac{\tilde{\sigma}_z^2}{\sigma_a^2}}\end{aligned}$$

where $\tilde{\sigma}_z^2 \equiv \frac{\lambda}{1-\lambda}\frac{\tau(1-\eta)+1-\frac{\lambda}{1-\lambda}}{\gamma\theta}\sigma_\epsilon^2$ represents sentiment volatility under a model without aggregate productivity shocks, only idiosyncratic productivity shocks $\epsilon_{j,t}$ and sentiments z_t.

Note that ψ_{za} consists of three parts: (1) $\frac{\tau\eta}{\gamma}$, its counterpart in the perfect information model (2) $\tilde{\sigma}_z^2$, sentiment volatility with only idiosyncratic productivity shocks, and (3) σ_a^2, volatility of productivity shocks.

Alternatively, rearranging[11] the terms of (58) and replacing $\psi_{za}^2\sigma_a^2$ with σ_z^2 gives a decomposition of σ_z^2, which is equivalent to (57):

$$\sigma_z^2 = \tilde{\sigma}_z^2 + \frac{\tau\eta}{\gamma}\psi_{za}\sigma_a^2 \tag{59}$$

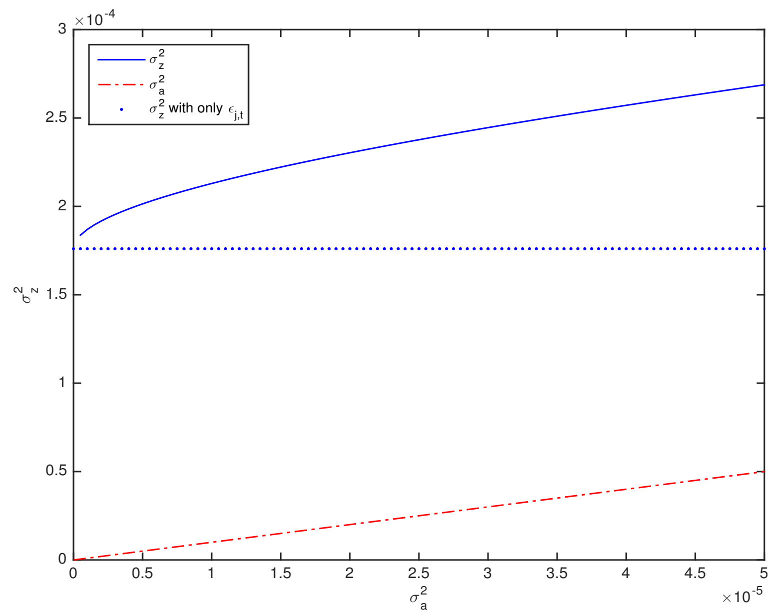
[10]Note that this is not an explicit function of σ_a^2, since ψ_{za} is still a function of σ_a^2.

[11]divide by γ, multiply by σ_a^2

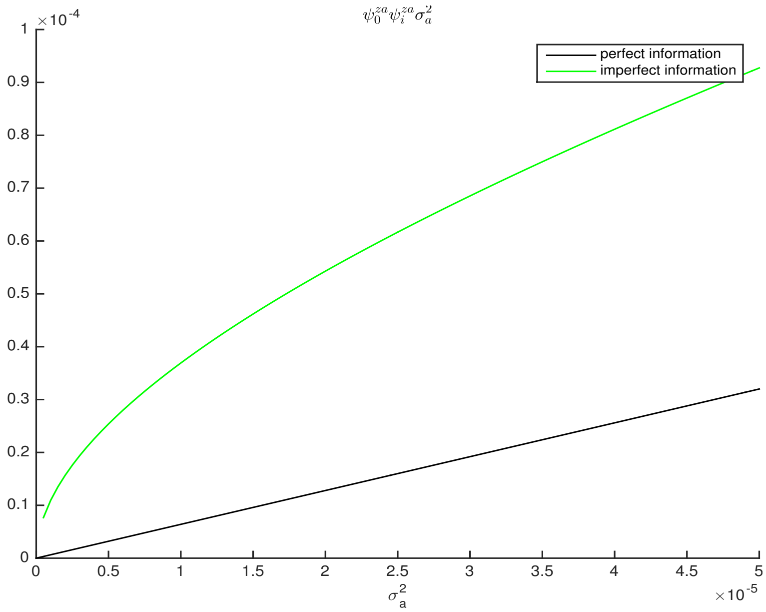
where $\tilde{\sigma}_z^2$ consists of sentiment volatility under idiosyncratic productivity shocks only ($\epsilon_{j,t}$), and $\frac{\tau\eta}{\gamma}\psi_{za}$ represents the transmission/amplification of σ_a^2. **Sentiments affect the way in which aggregate fundamental shocks affect aggregate output.** Note that as $\sigma_a^2 \to \infty$, $\frac{\tau\eta}{\gamma}\psi_{za} \to \left(\frac{\tau\eta}{\gamma}\right)^2$, which is equivalent to the transmission of productivity shocks to aggregate output under perfect information.[12]

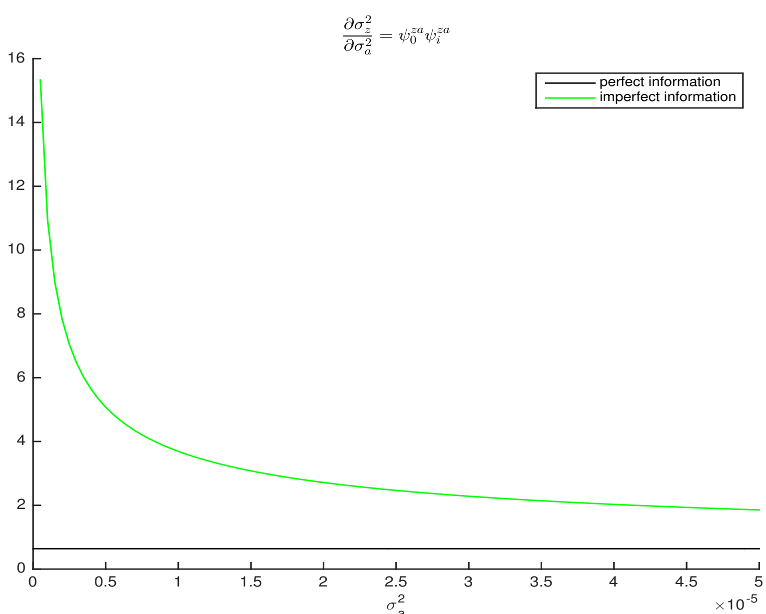
- **Considering the case of (+)**: it is always the case that $\psi_{za} > \frac{\tau\eta}{\gamma}$ and $\psi_{za} > 0$, then $\frac{\tau\eta}{\psi_{za}} < \gamma$. The smaller the term $\frac{\tilde{\sigma}_z^2}{\sigma_a^2}$, the larger is σ_z^2. As $\tilde{\sigma}_z^2$ is independent of σ_a^2, **σ_z^2 is increasing in σ_a^2.**

Next, decompose σ_z^2 into fundamental and non-fundamental components:



[12]Equation (59) is also consistent with the original conjecture $z_t = \psi_{za}a_t + \nu_z$, which implies $\sigma_z^2 = \psi_{za}^2\sigma_a^2$. (58) can be rearranged to show $\psi_{za}^2\sigma_a^2 = \tilde{\sigma}_z^2 + \frac{\tau\eta}{\gamma}\psi_{za}\sigma_a^2$.





where parameter calibrations are: $\gamma = 1$, $\theta = 6$, (less certain about these): $\lambda = 0.35$, $\tau = 2$, $\eta = 0.4$, $B = 0$, $\sigma_\epsilon^2 = 1.5$.
Comments:

1. The smaller σ_a^2, the larger the degree of amplification ($\frac{\partial \sigma_z^2}{\partial \sigma_a^2}$)

2. The larger σ_a^2, more the transmission of technology shocks approaches the perfect

information case (technology shocks become more salient): $\psi_{za} \to \frac{\tau\eta}{\gamma}$ and

$$\sigma_z^2 \to \tilde{\sigma}_z^2 + \left(\frac{\tau\eta}{\gamma}\right)^2 \sigma_a^2$$

With imperfect information, idiosyncratic productivity shocks have aggregate affects, as in the baseline. Under perfect information,

$$\sigma_z^2 = \left(\frac{\tau\eta}{\gamma}\right)^2 \sigma_a^2$$

6 Conclusion

In a model with minimal and plausible deviations from the standard New Keynesian model, namely, that firms must commit to prices and labor input before demand is realized, basing their decision on an endogenous signal that confounds idiosyncratic productivity/demand with aggregate demand, fluctuations in output can to be driven by non-fundamental sources. The set of equilibria is not policy invariant, which introduces a new tradeoff between stabilizing output and inflation. The Taylor principle is no longer sufficient to rule out non-fundamental equilibria. Although the policymaker can eliminate these alternative equilibria by relaxing its response to inflation, the coexistence of fundamental shocks to aggregate supply implies that the policymaker may not be able to respond optimally, in the sense that it can not stabilize the economy from the effects of fundamental shocks without allowing non-fundamental equilibria to arise. Although the case of price-setting firms is presented for comparison with the large literature on monetary policy in the presence of price rigidities, these results hold in the case of wage rigidities.

Future work should quantify the significance of non-fundamental fluctuations, and the welfare gains relative to a policy that is not robust to alternative equilibria. Introducing a combination of persistent sentiment, productivity, and demand shocks would yield a more interesting analysis of optimal monetary policy, particularly in the case where a central bank is unable to identify the source of fluctuations.

References

Adam, K. (2007). Optimal Monetary Policy with Imperfect Common Knowledge. *Journal of Monetary Economics*, 54(2):267–301.

Angeletos, G. and La'O, J. (2013a). Sentiments. *Econometrica*, 81(2):739–779.

Angeletos, G. and La'O, J. (2013b). Sentiments. *Econometrica*, 81(2):739–779.

Angeletos, G.-M., Hellwig, C., and Pavan, A. (2007). Dynamic Global Games of Regime Change: Learning, Multiplicity, and the Timing of Attacks. *Econometrica*, 75(3):711–756.

Angeletos, G.-M. and La'O, J. (2010). Noisy Business Cycles. *NBER Macroeconomics Annual*, 24(1):319–378.

Angeletos, G.-M. and Lian, C. (2016). Forward Guidance without Common Knowledge. Technical report, National Bureau of Economic Research, Cambridge, MA.

Angeletos, G.-M. and Werning, I. (2006). Crises and Prices: Information Aggregation, Multiplicity, and Volatility. *American Economic Review*, 96(5):1720–1736.

Benhabib, J., Wang, P., and Wen, Y. (2015). Sentiments and Aggregate Demand Fluctuations. *Econometrica*, 83(2):549–585.

Chahrour, R. and Gaballo, G. (2017). Learning from Prices: Amplification and Business Fluctuations.

Galí, J. (2015). *Monetary Policy, Inflation, and the Business Cycle*. Princeton University Press.

Hellwig, C., Mukherji, A., and Tsyvinski, A. (2006). Self-Fulfilling Currency Crises: The Role of Interest Rates. *American Economic Review*, 96(5):1769–1787.

Kasa, K. (2000). Forecasting the Forecasts of Others in the Frequency Domain. *Review of Economic Dynamics*, 3(4):726–756.

Lorenzoni, G. (2009). A Theory of Demand Shocks. *American Economic Review*, 99(5):2050–2084.

Melosi, L. (2016). Signalling Effects of Monetary Policy. *Review of Economic Studies*, 52(2-4).

Morris, S. and Shin, H. S. (2002). Social Value of Public Information. *American Economic Review*, 92(5):1521–1534.

Sargent, T. J. (1991). Equilibrium with Signal Extraction from Endogenous Variables. *Journal of Economic Dynamics and Control*, 15(2):245–273.

Townsend, R. M. (1983). Forecasting the Forecasts of Others. *Journal of Political Economy*, 91(4):546–588.

7 Appendix

7.1 Private signal correct up to *iid* noise

When agents actions are strategic substitutes, a private signal that conveys perfectly information needed for the agents' first order condition, but with iid noise, results in only the fundamental equilibrium. Consider the first order condition of a general beauty contest model, where a continuum of agents indexed by $j \in [0,1]$ take action conditional on a private signal $s_{j,t}$

$$y_{j,t} = \mathbb{E}[\underbrace{\beta_0 \varepsilon_{j,t} + \beta_1 y_t}_{x_{j,t}} | s_{j,t}]$$
$$s_{j,t} = \beta_0 \varepsilon_{j,t} + \beta_1 y_t + \nu_{j,t}$$

Note that $s_{j,t} = x_{j,t} + \nu_{j,t}$. Agent j's optimal response depends on an idiosyncratic iid shock $\varepsilon_{j,t} \sim N(0, \sigma^2_{\varepsilon_{j,t}})$, as well as on the aggregate response of other agents ($y_t = \int_0^1 y_{j,t} dj$), where $y_t \sim N(0, \sigma_y^2)$. The parameters β_0 and β_1 capture the elasticity of actions to the idiosyncratic shock and the aggregate variable. If $\beta_1 > 0$, agents face strategic complementarities. If $\beta_1 < 0$, agents face strategic substitutabilities.

Agent j's optimal response is

$$y_{j,t} = \frac{\beta_0^2 \sigma_\varepsilon^2 + \beta_1^2 \sigma_y^2}{\beta_0^2 \sigma_\varepsilon^2 + \beta_1^2 \sigma_y^2 + \sigma_\nu^2} (\beta_0 \varepsilon_{j,t} + \beta_1 y_t + \nu_{j,t})$$

As $\frac{\beta_0^2 \sigma_\varepsilon^2 + \beta_1^2 \sigma_y^2}{\beta_0^2 \sigma_\varepsilon^2 + \beta_1^2 \sigma_y^2 + \sigma_\nu^2} \in (0,1)$, we can only have sentiment driven equilibrium with this private signal if $\beta_1 > 1$.

However, if the private signal is instead $s_{j,t} = \lambda \varepsilon_{j,t} + (1-\lambda) y_t + \nu_{j,t}$, where $\lambda \neq \beta_0$ and $(1-\lambda) \neq \beta_1$, then

$$y_{j,t} = \frac{\beta_0 \lambda \sigma_\varepsilon^2 + \beta_1 (1-\lambda) \sigma_y^2}{\lambda^2 \sigma_\varepsilon^2 + (1-\lambda)^2 \sigma_y^2 + \sigma_\nu^2} (\lambda \varepsilon_{j,t} + (1-\lambda) y_t + \nu_{j,t})$$
$$y_t = \int_0^1 y_{j,t} dj = \frac{\beta_0 \lambda \sigma_\varepsilon^2 + \beta_1 (1-\lambda) \sigma_y^2}{\lambda^2 \sigma_\varepsilon^2 + (1-\lambda)^2 \sigma_y^2 + \sigma_\nu^2} (1-\lambda) y_t$$

In this case, any y_t is an equilibrium if

$$\frac{\beta_0 \lambda \sigma_\varepsilon^2 + \beta_1 (1-\lambda) \sigma_y^2}{\lambda^2 \sigma_\varepsilon^2 + (1-\lambda)^2 \sigma_y^2 + \sigma_\nu^2} (1-\lambda) = 1$$

The volatility of y_t is determined by parameters of the model.

$$\sigma_y^2 = \frac{\beta_0 \lambda (1-\lambda) - \lambda^2}{(1-\lambda)^2 (1-\beta_1)} \sigma_\varepsilon^2 - \frac{1}{(1-\lambda)^2 (1-\beta_1)} \sigma_\nu^2$$

The private signal that is correct up to *iid* noise allows firms to respond to the two shocks in the correct proportions. In order for sentiment driven equilibria to exist when firms' actions are strategic substitutes, information frictions must be such that firms misattribute some of the sentiment component in their signal to idiosyncratic preference for their good.

7.2 Expected future inflation with *iid* shock to sentiments

Let lower-case variables with a hat symbol represent variables in log-deviation from steady state. If z_t is *iid* and with mean equal to z, and if we conjecture $\hat{y}_t = \hat{c}_t = \hat{z}_t$, then $\forall\ k \geq 1$,

$$\mathbb{E}_t \hat{c}_{t+k} = 0 \tag{60}$$

$$\mathbb{E}_t \hat{y}_{t+k} = 0 \tag{61}$$

- **Motivation**
  Following (60), it can be shown that

$$\mathbb{E}_t \hat{\pi}_{t+1} = 0$$
$$\mathbb{E}_t p_{t+1} = p_t$$

- **Real interest rate path as a function of *iid* shock z_t**
  The Euler equation in period $t + k$:

$$\begin{aligned} \hat{c}_{t+k} &= \mathbb{E}_{t+k} \hat{c}_{t+k+1} - \frac{1}{\gamma}[i_{t+k} - \mathbb{E}_{t+k} \hat{\pi}_{t+k+1} - \rho] \\ &= \mathbb{E}_{t+k} \hat{c}_{t+k+1} - \frac{1}{\gamma}[r_{t+k} - \rho] \\ &= \mathbb{E}_{t+k} \hat{c}_{t+k+1} - \frac{1}{\gamma} \hat{r}_{t+k} \end{aligned}$$

  where $\rho \equiv log(\frac{1}{\beta})$ and the real interest rate $r_t \equiv i_t - \mathbb{E}_t \pi_{t+1}$. Note that under the assumption of zero inflation in steady state, ρ is both the steady state nominal interest rate and steady state real interest rate. Taking the expectation at time t of both sides and applying the law of iterated expectations,

$$\mathbb{E}_t \hat{c}_{t+k} = \mathbb{E}_t \hat{c}_{t+k+1} - \frac{1}{\gamma} \mathbb{E}_t \hat{r}_{t+k}$$

  Using (60), $\forall\ k \geq 1$

$$\mathbb{E}_t \hat{r}_{t+k} = 0 \tag{62}$$

- **Inflation in terms of real interest rate path**
  Next, use Fisher equation $r_t = i_t - \mathbb{E}_t \pi_{t+1}$ to show that $\mathbb{E}_t \hat{\pi}_{t+1} = 0$. Combining

these two expressions gives inflation (and hence the price level) as a function of the path of the real interest rate. Again, under the assumption of zero inflation in the steady state, the Fisher equation is

$$r_t = i_t - \mathbb{E}_t \hat{\pi}_{t+1}$$

Assume the central bank follows the Taylor rule given by

$$i_t = \rho + \phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t$$

$$\begin{aligned} r_t &= i_t - \mathbb{E}_t \hat{\pi}_{t+1} \\ &= \rho + \phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t - \mathbb{E}_t \hat{\pi}_{t+1} \\ \hat{\pi}_t &= \frac{1}{\phi_\pi} [\hat{r}_t - \phi_y \hat{y}_t + \mathbb{E}_t \hat{\pi}_{t+1}] \end{aligned}$$

Iterating forwards and using (61):

$$\hat{\pi}_t = \sum_{k=0}^{\infty} \frac{1}{\phi_\pi^{k+1}} \mathbb{E}_t \hat{r}_{t+k} - \sum_{k=0}^{\infty} \left(\frac{\phi_y}{\phi_\pi}\right)^{k+1} \mathbb{E}_t \hat{y}_{t+k}$$

Then at $t+1$, we will have

$$\hat{\pi}_{t+1} = \sum_{k=0}^{\infty} \frac{1}{\phi_\pi^{k+1}} \mathbb{E}_{t+1} \hat{r}_{t+k+1} - \sum_{k=0}^{\infty} \left(\frac{\phi_y}{\phi_\pi}\right)^{k+1} \mathbb{E}_{t+1} \hat{y}_{t+k+1}$$

Taking the expectation at time t of both sides, and applying the law of iterated expectations:

$$\mathbb{E}_t \hat{\pi}_{t+1} = \sum_{k=0}^{\infty} \frac{1}{\phi_\pi^{k+1}} \mathbb{E}_t \hat{r}_{t+k+1} - \sum_{k=0}^{\infty} \left(\frac{\phi_y}{\phi_\pi}\right)^{k+1} \mathbb{E}_t \hat{y}_{t+k+1}$$

Using (62) and (61):

$$\mathbb{E}_t \hat{\pi}_{t+1} = 0$$

7.3 Optimal Allocation of Consumption Expenditures

As shown in Galí (2015): Firms produce a differentiated good for which it sets the price. C_t is now a consumption index:

$$C_t \equiv \left[\int_0^1 \epsilon_{j,t}^{\frac{1}{\theta}} C_{j,t}^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}}$$

with $C_{j,t}$ as the quantity of good j consumed by the household in period t. The aggregate price index is given by

$$P_t \equiv \left[\int_0^1 \epsilon_{j,t} P_{j,t}^{1-\theta} dj\right]^{\frac{1}{1-\theta}}$$

- Maximizing C_t with respect to any expenditure level: ($\int_0^1 P_{j,t}C_{j,t}dj \equiv Z_t$):

$$\mathbb{L} = \left[\int_0^1 \epsilon_{j,t}^{\frac{1}{\theta}} C_{j,t}^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} - \lambda \left[\int_0^1 P_{j,t}C_{j,t}dj - Z_t\right]$$

Taking the first order condition with respect to $C_{j,t}$:

$$\frac{\theta}{\theta-1}\left[\int_0^1 \epsilon_{j,t}^{\frac{1}{\theta}} C_{j,t}^{\frac{\theta-1}{\theta}} dj\right]^{\frac{1}{\theta-1}} C_{j,t}^{-\frac{1}{\theta}} = \lambda P_{j,t}$$

This implies that for any two goods, i and j:

$$\frac{\theta}{\theta-1}\epsilon_{j,t}^{\frac{1}{\theta}} C_t^{\frac{1}{\theta}} C_{j,t}^{-\frac{1}{\theta}} = \lambda P_{j,t}$$
$$\frac{\theta}{\theta-1}\epsilon_{i,t}^{\frac{1}{\theta}} C_t^{\frac{1}{\theta}} C_{i,t}^{-\frac{1}{\theta}} = \lambda P_{i,t}$$

- Taking the ratio of optimal consumption demand for goods i and j:

$$C_{j,t} = C_{i,t}\left(\frac{P_{j,t}}{P_{i,t}}\right)^{-\theta}\left(\frac{\epsilon_{j,t}}{\epsilon_{i,t}}\right)$$

- Substituting this expression into expenditure equation ($\int_0^1 P_{j,t}C_{j,t}dj \equiv Z_t$)

$$\int_0^1 P_{j,t}\left[C_{i,t}\left(\frac{P_{j,t}}{P_{i,t}}\right)^{-\theta}\left(\frac{\epsilon_{j,t}}{\epsilon_{i,t}}\right)\right]dj = Z_t$$
$$C_{i,t}\epsilon_{i,t}^{-1}\underbrace{\int_0^1 P_{j,t}^{1-\theta}\epsilon_{j,t}dj}_{P_t^{1-\theta}} = P_{i,t}^{-\theta}Z_t$$
$$C_{i,t}\epsilon_{i,t}^{-1} = \left(\frac{P_{i,t}}{P_t}\right)^{-\theta}\left(\frac{Z_t}{P_t}\right)$$

- Substituting $Z_t \equiv \int_0^1 P_{j,t}C_{j,t}dj = P_tC_t$:

$$C_{i,t} = \left(\frac{P_t}{P_{i,t}}\right)^{\theta} C_t\epsilon_{i,t}$$

7.4 Sentiment-driven equilibrium steady state

As shown in Benhabib et al. (2015):

- First, express $y_{j,t}$ as a function of the fundamentals $(\varepsilon_{j,t}, z_t)$ The firm's optimal production, incorporating households' optimal labor supply decision (96), and contingent on signal $s_{j,t}$ is:

$$Y_{j,t} = \left[\left(1 - \frac{1}{\theta}\right)\frac{A}{\Psi}\mathbb{E}_t[\epsilon_{j,t}^{\frac{1}{\theta}} Y_t^{\frac{1}{\theta}-1} | s_{j,t}]\right]^{\theta}$$

Let $\varepsilon_{j,t} \equiv \log \epsilon_{j,t} \sim N(0, \sigma_\varepsilon^2)$ and $z_t \equiv (\log Z_t) - \phi_0 \sim N(0, \sigma_z^2)$, firm j's signal is

$$s_{j,t} = \lambda \varepsilon_{j,t} + (1-\lambda) z_t$$

Without loss of generality, normalize $\left(1 - \frac{1}{\theta}\right)\frac{A}{\Psi}$ to 1. Firm production is then:

$$Y_{j,t} = \left(\mathbb{E}_t[\epsilon_{j,t}^{\frac{1}{\theta}} Y_t^{\frac{1}{\theta}-1} | s_{j,t}]\right)^{\theta}$$

Define $y_t \equiv (\log Y_t) - \phi_0$. Unless specified otherwise, let lower-case letters represent the variable in logs. In this equilibrium, as aggregate demand is sentiment driven, we can replace y_t in the firm's response with z_t:

$$y_{j,t} = (1-\theta)\phi_0 + \theta \log \mathbb{E}_t \left[\exp\left(\frac{1}{\theta}\varepsilon_{j,t} + \frac{1-\theta}{\theta} z_t\right) | s_{j,t}\right]$$

To compute the conditional expectation, note that $\mathbb{E}_t \left[\exp\left(\frac{1}{\theta}\varepsilon_{j,t} + \frac{1-\theta}{\theta} z_t\right) | s_{j,t}\right]$ is the moment generating function of normal random variable $\left(\frac{1}{\theta}\varepsilon_{j,t} + \frac{1-\theta}{\theta} z_t\right) | s_{j,t}$. Then

$$\mathbb{E}_t \left[\exp\left(\frac{1}{\theta}\varepsilon_{j,t} + \frac{1-\theta}{\theta} z_t\right) | s_{j,t}\right] = \exp\left[\mathbb{E}_t\left(\frac{1}{\theta}\varepsilon_{j,t} + \frac{1-\theta}{\theta} z_t | s_{j,t}\right) + \frac{1}{2}\mathrm{Var}\left(\frac{1}{\theta}\varepsilon_{j,t} + \frac{1-\theta}{\theta} z_t | s_{j,t}\right)\right]$$

where

$$\mathbb{E}_t\left(\frac{1}{\theta}\varepsilon_{j,t} + \frac{1-\theta}{\theta} z_t | s_{j,t}\right) = \frac{\mathrm{cov}(\frac{1}{\theta}\varepsilon_{j,t} + \frac{1-\theta}{\theta} z_t, s_{j,t})}{\mathrm{var}(s_{j,t})} s_{j,t} \tag{63}$$

$$= \frac{\frac{1}{\theta}\lambda\sigma_\varepsilon^2 + \frac{1-\theta}{\theta}(1-\lambda)\sigma_z^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2}(\lambda\varepsilon_{j,t} + (1-\lambda)z_t) \tag{64}$$

For now, let $\Omega_s \equiv \mathrm{Var}\left(\frac{1}{\theta}\varepsilon_{j,t} + \frac{1-\theta}{\theta} z_t | s_{j,t}\right)$. As $\frac{1}{\theta}\varepsilon_{j,t}, \frac{1-\theta}{\theta} z_t$ are Gaussian, Ω_s does not depend on $s_{j,t}$.

$$y_{j,t} = (1-\theta)\phi_0 + \theta \frac{\frac{1}{\theta}\lambda\sigma_\varepsilon^2 + \frac{1-\theta}{\theta}(1-\lambda)\sigma_z^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2}(\lambda\varepsilon_{j,t} + (1-\lambda)z_t) + \frac{\theta}{2}\Omega_s \tag{65}$$

$$\equiv \varphi_0 + \theta\mu(\lambda\varepsilon_{j,t} + (1-\lambda)z_t) \tag{66}$$

where

$$\mu = \frac{\frac{1}{\theta}\lambda\sigma_\varepsilon^2 + \frac{1-\theta}{\theta}(1-\lambda)\sigma_z^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2} \tag{67}$$

$$\varphi_0 = (1-\theta)\phi_0 + \frac{\theta}{2}\Omega_s \tag{68}$$

- Next, we find an expression for y_t in terms of $\varepsilon_{j,t}$ and z_t:
  Using equilibrium condition (102) which equates aggregate demand and aggregate supply, get an expression for y_t in terms of $y_{j,t}$:

$$\left(1-\frac{1}{\theta}\right)\log Y_t = \log\left(\int \epsilon_{j,t}^{\frac{1}{\theta}} Y_{j,t}^{\frac{\theta-1}{\theta}}\, dj\right)$$

$$\left(1-\frac{1}{\theta}\right)(\phi_0+z_t) = \log \mathbb{E}_t\left(\epsilon_{j,t}^{\frac{1}{\theta}} Y_{j,t}^{\frac{\theta-1}{\theta}}\right)$$

$$= \log \mathbb{E}_t\left(\exp\left[\frac{1}{\theta}\varepsilon_{j,t} + \frac{\theta-1}{\theta} y_{j,t}\right]\right)$$

  Replacing $y_{j,t}$ with (66) and using the properties of a moment generating function for normal random variable $\left[\frac{1}{\theta}\varepsilon_{j,t} + \frac{\theta-1}{\theta}\left[\varphi_0 + \theta\mu(\lambda\varepsilon_{j,t} + (1-\lambda)z_t)\right]\right]$

$$\left(1-\frac{1}{\theta}\right)(\phi_0+z_t) = \log \mathbb{E}_t\left(\exp\left[\frac{1}{\theta}\varepsilon_{j,t} + \frac{\theta-1}{\theta}\left[\varphi_0 + \theta\mu(\lambda\varepsilon_{j,t} + (1-\lambda)z_t)\right]\right]\right) \tag{69}$$

$$= \left(1-\frac{1}{\theta}\right)\varphi_0 + \left[\frac{\theta-1}{\theta}\theta\mu(1-\lambda)\right] z_t + \frac{1}{2}\left[\frac{1}{\theta} + \frac{\theta-1}{\theta}\theta\mu\lambda\right]^2 \sigma_\varepsilon^2 \tag{70}$$

$$\left(\frac{\theta-1}{\theta}\right)(\phi_0+z_t) = \frac{\theta-1}{\theta}\varphi_0 + \frac{\theta-1}{\theta}\theta\mu(1-\lambda)z_t + \frac{1}{2}\left(\frac{1}{\theta} + \frac{\theta-1}{\theta}\theta\mu\lambda\right)^2 \sigma_\varepsilon^2 \tag{71}$$

- Match the coefficients in (71) to get two constraints for the parameters to be determined: ϕ_0, σ_z^2:

$$\theta\mu = \frac{1}{1-\lambda} \tag{72}$$

$$\frac{\theta-1}{\theta}\phi_0 = \frac{\theta-1}{\theta}\varphi_0 + \frac{1}{2}\left(\frac{1}{\theta} + \frac{\theta-1}{\theta}\theta\mu\lambda\right)^2 \sigma_\varepsilon^2 \tag{73}$$

- Solving for σ_z^2
  σ_z^2 can be solved for in terms of the structural parameters using using the first constraint and (67)

$$\sigma_z^2 = \frac{\lambda(1-2\lambda)}{(1-\lambda)^2\theta}\sigma_\varepsilon^2 \tag{74}$$

  A more intuitive expression:

$$\sigma_z^2 = \frac{\lambda}{1-\lambda}\frac{1-\frac{\lambda}{1-\lambda}}{\theta}\sigma_\epsilon^2$$

- Solving for ϕ_0
  From (71):

$$\phi_0 = \varphi_0 + \frac{1}{2}\frac{\theta-1}{\theta}\left[\frac{1}{\theta-1} + \frac{\lambda}{1-\lambda}\right]^2 \sigma_\epsilon^2$$

Substituting for φ_0 and simplifying,

$$\phi_0 = \frac{\Omega_s}{2} - \log \psi + \frac{1}{2\theta}\frac{\theta-1}{\theta}\left[\frac{1}{\theta-1} + \frac{\lambda}{1-\lambda}\right]^2 \sigma_\epsilon^2$$

The conditional variance, Ω_s, can be decomposed:

$$\begin{aligned}
\Omega_s &\equiv \text{var}\left(\frac{1}{\theta}\varepsilon_{j,t} + \frac{1-\theta}{\theta}z_t | s_{j,t}\right) \\
&= \text{var}(\frac{1}{\theta}\varepsilon_{j,t} + \frac{1-\theta}{\theta}z_t) - \frac{[\text{cov}(\frac{1}{\theta}\varepsilon_{j,t} + \frac{1-\theta}{\theta}z_t, s_{j,t})]^2}{\text{var}(s_{j,t})} \\
\Omega_s &= \left(\frac{1}{\theta}\right)^2 \sigma_\varepsilon^2 + \left(\frac{1-\theta}{\theta}\right)^2 \sigma_z^2 - \mu\left[\frac{1}{\theta}\lambda\sigma_\varepsilon^2 + \frac{1-\theta}{\theta}(1-\lambda)\sigma_z^2\right] \\
&= \left(\frac{1}{\theta}\right)^2 \sigma_\varepsilon^2 + \left(\frac{1-\theta}{\theta}\right)^2 \sigma_z^2 - \left(\frac{1}{\theta}\frac{1}{1-\lambda}\right)\left[\frac{1}{\theta}\lambda\sigma_\varepsilon^2 + \frac{1-\theta}{\theta}(1-\lambda)\sigma_z^2\right] \\
&= \frac{1}{\theta^2}\left(1 - \frac{\lambda}{1-\lambda}\right)\sigma_\epsilon^2 + \frac{1-\theta}{\theta^2}(-\theta\sigma_z^2)
\end{aligned}$$

where the third equality uses (63) and (67). Incorporating (74),

$$\Omega_s = \frac{1}{\theta^2}\left(1 - \frac{\lambda}{1-\lambda}\right)\left(1 + (1-\theta)\left(-\frac{\lambda}{1-\lambda}\right)\right)\sigma_\epsilon^2$$

Simplifying,

$$\Omega_s = \frac{(1-\lambda)(1-2\lambda) + (\theta-1)\lambda(1-2\lambda)}{\theta^2(1-\lambda)^2}\sigma_\varepsilon^2$$

Then by (68) and (73),

$$\phi_0 = \frac{(1-\lambda)(\theta-1)\lambda}{\theta(1-\lambda)} \underbrace{\frac{1}{2(\theta-1)}\sigma_\varepsilon^2}_{\phi_0^*}$$

where ϕ_0^* denotes the steady state of the fundamental equilibrium (See section (7.12.1)).

7.5 Proof of Proposition 1

In a sentiment driven equilibrium with price-setting firms, aggregate demand may be driven by sentiments. In a self-fulfilling equilibrium, $Y_t = Z_t$. To find the volatility of output and its mean in this equilibrium,

- First, find an expression for $\log P_{j,t}$ in terms of the fundamentals, $\log \epsilon_{j,t}$ and $\log Y_t$. From (9),

$$P_{j,t} = \left(\frac{\theta}{\theta-1}\right)\Psi\frac{\mathbb{E}[\epsilon_{j,t}^{1-\tau}Y_t^{\sigma+1}|s_{j,t}]}{\mathbb{E}[\epsilon_{j,t}Y_t|s_{j,t}]}$$

Without loss of generality, normalize $\frac{\theta}{\theta-1}\Psi$ to 1. Taking the log of this expression,

$$p_{j,t} = \log \mathbb{E}_t[Y_t^{\sigma+1}\epsilon_{j,t}^{1-\tau}|s_{j,t}] - \log \mathbb{E}_t[\epsilon_{j,t}Y_t|s_{j,t}]$$

Using the properties of a moment generating function for a normal random variable, the first term can be expressed as

$$\log \mathbb{E}_t[Y_t^{\sigma+1}\epsilon_{j,t}^{1-\tau}|s_{j,t}] = \log \mathbb{E}_t[e^{(\sigma+1)(y_t+\phi_0)+(1-\tau)\varepsilon_{j,t}}|s_{j,t}] \tag{75}$$

$$= (\sigma+1)\phi_0 + \mathbb{E}_t[(\sigma+1)y_t + (1-\tau)\varepsilon_{j,t}|s_{j,t}] + \frac{1}{2}\underbrace{\mathrm{Var}[(\sigma+1)y_t + (1-\tau)\varepsilon_{j,t}|s_{j,t}]}_{\Omega_1} \tag{76}$$

$$= (\sigma+1)\phi_0 + \underbrace{\frac{(\sigma+1)(1-\lambda)\sigma_z^2 + (1-\tau)\lambda\sigma_\varepsilon^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2}}_{\mu_1} s_{j,t} + \frac{1}{2}\Omega_1 \tag{77}$$

$$= (\sigma+1)\phi_0 + \mu_1 s_{j,t} + \frac{1}{2}\Omega_1 \tag{78}$$

Similarly, the second term can be expressed as:

$$\log \mathbb{E}_t[\epsilon_{j,t}Y_t|s_{j,t}] = \log \mathbb{E}_t[e^{\varepsilon_{j,t}+y_t+\phi_0}|s_{j,t}] \tag{79}$$

$$= \phi_0 + \mathbb{E}_t[\varepsilon_{j,t} + y_t|s_{j,t}] + \frac{1}{2}\underbrace{\mathrm{Var}[\varepsilon_{j,t} + y_t|s_{j,t}]}_{\Omega_2} \tag{80}$$

$$= \phi_0 + \underbrace{\frac{(1-\lambda)\sigma_z^2 + \lambda\sigma_\varepsilon^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2}}_{\mu_2} s_{j,t} + \frac{1}{2}\Omega_2 \tag{81}$$

$$= \phi_0 + \mu_2 s_{j,t} + \frac{1}{2}\Omega_2 \tag{82}$$

Then

$$p_{j,t} = \underbrace{\sigma\phi_0 + \frac{1}{2}(\Omega_1 - \Omega_2)}_{\varphi_0} + \underbrace{\frac{\sigma(1-\lambda)\sigma_z^2 - \tau\lambda\sigma_\varepsilon^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2}}_{\bar{\mu}\equiv\mu_1-\mu_2} s_{j,t} \tag{83}$$

$$= \varphi_0 + \bar{\mu}(\lambda\varepsilon_{j,t} + (1-\lambda)z_t) \tag{84}$$

- Next, substitute (84) into the aggregate price index and use the normalization of $P_t = 1$ to solve for φ_0 and σ_z^2. Taking the log of (8),

$$\begin{aligned}(1-\theta)p_t &= \log \mathbb{E}[\epsilon_{j,t}P_{j,t}^{1-\theta}] \\ &= \log \mathbb{E}[e^{\varepsilon_{j,t}+(1-\theta)p_{j,t}}] \\ &= \log \mathbb{E}[e^{\varepsilon_{j,t}+(1-\theta)(\varphi_0+\bar{\mu}(\lambda\varepsilon_{j,t}+(1-\lambda)z_t))}]\end{aligned}$$

By the properties of the moment generating function for normally distributed variables,

$$
\begin{aligned}
(1-\theta)p_t &= (1-\theta)\varphi_0 + \frac{1}{2}\mathrm{Var}([1+(1-\theta)\bar{\mu}\lambda]\varepsilon_{j,t}) + (1-\theta)\bar{\mu}(1-\lambda)z_t \\
&= (1-\theta)\varphi_0 + \frac{[1+(1-\theta)\bar{\mu}\lambda]^2}{2}\sigma_\varepsilon^2 + (1-\theta)\bar{\mu}(1-\lambda)z_t \\
p_t &= \varphi_0 + \frac{[1+(1-\theta)\bar{\mu}\lambda]^2}{2(1-\theta)}\sigma_\varepsilon^2 + \bar{\mu}(1-\lambda)z_t
\end{aligned}
$$

As P_t is normalized to 1, $p_t \equiv \log P_t = 0$,

$$
0 = \varphi_0 + \frac{[1+(1-\theta)\bar{\mu}\lambda]^2}{2(1-\theta)}\sigma_\varepsilon^2 + \bar{\mu}(1-\lambda)z_t \tag{85}
$$

Two constraints result from equating the coefficients in (85):

$$
\begin{aligned}
\bar{\mu}(1-\lambda) &= 0 \\
\varphi_0 + \frac{[1+(1-\theta)\bar{\mu}\lambda]^2}{2(1-\theta)}\sigma_\varepsilon^2 &= 0
\end{aligned}
$$

The first constraint implies $\bar{\mu} = 0$, since $\theta > 1$ and $\lambda \in (0,1)$. Then by (84),

$$
\sigma_z^2 = \frac{\lambda}{1-\lambda}\frac{\tau}{\sigma}\sigma_\varepsilon^2 \tag{86}
$$

From the second constraint, using $\bar{\mu} = 0$,

$$
\varphi_0 = \frac{1}{2(\theta-1)}\sigma_\varepsilon^2 \tag{87}
$$

- Finally, (86) and (87) can be used to find the steady state of the sentiment-driven equilibrium (ϕ_0). It can be shown that this steady state is lower than that of the fundamental equilibrium. Rearranging the terms in (84), where φ_0 was initially defined,

$$
\phi_0 = \frac{1}{\sigma}\left[\varphi_0 - \frac{1}{2}(\Omega_1 - \Omega_2)\right] \tag{88}
$$

  - In (76), $\Omega_1 \equiv \mathrm{Var}[(\sigma+1)y_t + (1-\tau)\varepsilon_{j,t}|s_{j,t}]$. The conditional variance of a normally distributed random variable can be decomposed as

$$
\begin{aligned}
\Omega_1 &= \mathrm{Var}[(\sigma+1)y_t + (1-\tau)\varepsilon_{j,t}] - \frac{(\mathrm{cov}[(\sigma+1)y_t + (1-\tau)\varepsilon_{j,t}, s_{j,t}])^2}{\mathrm{Var}(s_{j,t})} \\
&= (\sigma+1)^2\sigma_z^2 + (1-\tau)^2\sigma_\varepsilon^2 - \mu_1(\mathrm{cov}[(\sigma+1)y_t + (1-\tau)\varepsilon_{j,t}, s_{j,t}]) \\
&= (\sigma+1)^2\sigma_z^2 + (1-\tau)^2\sigma_\varepsilon^2 - \mu_1[(\sigma+1)(1-\lambda)\sigma_z^2 + (1-\tau)\lambda\sigma_\varepsilon^2]
\end{aligned}
$$

  where μ_1 is defined in (77). Substituting σ_z^2 with (86),

$$
\Omega_1 = (\sigma+1)^2\sigma_z^2 + (1-\tau)^2\sigma_\varepsilon^2 - \mu_1\frac{\lambda(\tau+\sigma)}{\sigma}\sigma_\varepsilon^2
$$

– By the same procedure, $\Omega_2 \equiv \text{Var}[\varepsilon_{j,t} + y_t | s_{j,t}]$ is equivalent to

$$\begin{aligned}
\Omega_2 &= \text{Var}[y_t + \varepsilon_{j,t}] - \frac{(\text{cov}[y_t + \varepsilon_{j,t}, s_{j,t}])^2}{\text{Var}(s_{j,t})} \\
&= \sigma_\varepsilon^2 + \sigma_z^2 - \mu_2(\text{cov}[\varepsilon_{j,t} + z_t, s_{j,t}]) \\
&= \sigma_\varepsilon^2 + \sigma_z^2 - \mu_2 \frac{\lambda(\tau + \sigma)}{\sigma} \sigma_\varepsilon^2
\end{aligned}$$

where μ_2 is defined in (81).

Then, substituting φ_0 with (87) in (88), ϕ_0 can be expressed as

$$\begin{aligned}
\phi_0 &= \frac{1}{\sigma}\left[\frac{1}{2(\theta-1)}\sigma_\varepsilon^2 - \frac{1}{2}(\Omega_1 - \Omega_2)\right] \\
&= \frac{1}{\sigma}\left[\frac{1}{2(\theta-1)}\sigma_\varepsilon^2 - \frac{1}{2}\left([(\sigma+1)^2 - 1]\sigma_z^2 + [(1-\tau)^2 - 1]\sigma_\varepsilon^2 - \frac{\lambda(\tau+\sigma)}{\sigma}(\mu_1 - \mu_2)\sigma_\varepsilon^2\right)\right]
\end{aligned}$$

Note that equating coefficients in (85) implies that $\bar{\mu} \equiv \mu_1 - \mu_2 = 0$,

$$\begin{aligned}
\phi_0 &= \frac{1}{\sigma}\left[\frac{1}{2(\theta-1)}\sigma_\varepsilon^2 - \frac{1}{2}\left([(\sigma+1)^2 - 1]\sigma_z^2 + [(1-\tau)^2 - 1]\sigma_\varepsilon^2\right)\right] \\
&= \frac{1}{\sigma}\left[\frac{1}{2(\theta-1)}\sigma_\varepsilon^2 - \frac{1}{2}\tau\left(\frac{\lambda}{1-\lambda}[\sigma+2] + [\tau-2]\right)\sigma_\varepsilon^2\right] \\
&= \frac{1}{2(\theta-1)}\frac{1}{\sigma}\left[1 - \tau(\theta-1)\left(\frac{\lambda}{1-\lambda}[\sigma+2] + [\tau-2]\right)\right]\sigma_\varepsilon^2
\end{aligned}$$

- Finally, it can be shown that the steady state of output in the imperfect information case is less than its counterpart in the perfect information case ($\phi_0 < \phi_0^*$), where ϕ_0^* is specified in (22). Note that $\phi_0 < \phi_0^*$ if

$$1 - \tau(\theta-1)\left(\frac{\lambda}{1-\lambda}[\sigma+2] + [\tau-2]\right) < [1 + \tau(\theta-1)]^2$$

Using the fact that $\theta > 1$, $\tau > 0$, $\lambda \in (0,1)$, the above inequality is true if

$$\tau > -\theta(\sigma+2)\frac{\lambda}{1-\lambda}$$

or alternatively,

$$\sigma > -\left[\frac{\tau(1-\lambda)}{\theta\lambda} + 2\right]$$

7.6 Equilibrium, Calvo wage setting

The equilibrium is characterized by the Euler equation, the New Keynesian Phillips curve for wage inflation, the firms' optimal production function, the central bank's policy rule, market clearing, and the real wage identity:

$$
\begin{aligned}
\hat{c}_t &= \mathbb{E}_t \hat{c}_{t+1} - \frac{1}{\sigma}(i_t - \rho - \mathbb{E}_t \hat{\pi}_{t+1}) \\
\pi_t^w &= \beta \mathbb{E}_t \pi_{t+1}^w - \lambda_w \hat{\mu}_t^w \\
\hat{y}_{j,t} &= \mathbb{E}_t(\hat{\varepsilon}_{j,t} + \hat{y}_t - \theta r\hat{w}_t | s_{j,t}) \\
i_t &= \rho + \phi_\pi^w \hat{\pi}_t^w + \phi_y \hat{y}_t \\
\hat{y}_t &= \hat{c}_t \\
r\hat{w}_{t+1} &= r\hat{w}_t + \mathbb{E}_t \hat{\pi}_{t+1}^w - \mathbb{E}_t \hat{\pi}_{t+1}
\end{aligned}
$$

Also, households' beliefs about aggregate output are correct

$$\hat{y}_t = \hat{z}_t$$

To find $r\hat{w}_t, \hat{\pi}_t, \hat{\pi}_t^w$ in terms of $\hat{z}_t$, guess and verify the following policy functions:

$$
\begin{aligned}
\hat{c}_t &= \hat{z}_t \\
r\hat{w}_t &= a_r \hat{z}_t + b_r r\hat{w}_{t-1} \\
\pi_t^w &= a_w \hat{z}_t + b_w r\hat{w}_{t-1} \\
\pi_t &= a_p \hat{z}_t + b_p r\hat{w}_{t-1}
\end{aligned}
$$

7.7 Robustness of results to alternative preferences

7.7.1 Non-linear disutility of labor, firm sets quantity

Consider a more general utility function for households that is non-linear in labor supply.

Households choose labor supply (N_t) to maximize utility

$$\max_{N_t} \frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\varphi}}{1+\varphi}$$

subject to budget constraint

$$P_t C_t \leq W_t N_t + \Pi_t$$

The resulting first order condition,

$$
\begin{aligned}
\frac{-U_n}{U_c} &= \frac{W_t}{P_t} \\
C_t^\gamma N_t^\varphi &= \frac{W_t}{P_t}
\end{aligned}
$$

This implies that the price level is as follows:

$$P_t = \frac{W_t}{C_t^\gamma N_t^\varphi}$$

Substituting N_t with the production function $Y_t = AN_t$ and applying the market clearing condition, $Y_t = C_t$,

$$P_t = \frac{W_t}{C_t^{\gamma+\varphi}} A^{\varphi} \tag{89}$$

From (101) The firms' first order condition is

$$Y_{j,t} = \left[\left(1 - \frac{1}{\theta}\right) A\mathbb{E}_t \left[(\epsilon_{j,t} Y_t)^{\frac{1}{\theta}} \frac{P_t}{W_t} | s_{j,t} \right] \right]^{\theta}$$

Substituting P_t with (89),

$$Y_{j,t} = \left[\left(1 - \frac{1}{\theta}\right) A^{1+\varphi} \mathbb{E}_t \left[\epsilon_{j,t}^{\frac{1}{\theta}} Y_t^{\frac{1}{\theta} - \gamma + \varphi} | s_{j,t} \right] \right]^{\theta}$$

Alternatively, substituting the real wage with the household's optimal labor supply condition,

$$Y_{j,t}^{\frac{1}{\theta}} = \left[\left(1 - \frac{1}{\theta}\right) A\mathbb{E}_t \left[\epsilon_{j,t}^{\frac{1}{\theta}} Y_t^{\frac{1}{\theta} - \gamma} N_t^{-\varphi} | s_{j,t} \right] \right]$$

Replacing $N_t = \int N_{j,t} dj = \int \frac{Y_{j,t}}{A} dj$

$$Y_{j,t}^{\frac{1}{\theta}} = \left[\left(1 - \frac{1}{\theta}\right) A\mathbb{E}_t \left[\epsilon_{j,t}^{\frac{1}{\theta}} Y_t^{\frac{1}{\theta} - \gamma} \left(\int \frac{Y_{j,t}}{A} dj \right)^{-\varphi} | s_{j,t} \right] \right]$$

Key steps:

(a) Guess $y_{j,t} = D + Bs_{j,t}$. Equating coefficients:

$$D = \frac{1}{1 + \varphi\theta} \left((1 - \gamma\theta)\phi_0 - \varphi\theta \left[\log \frac{1}{A} + \frac{(B\lambda)^2}{2} \sigma_\epsilon^2 \right] + \frac{\theta}{2} \Omega_s \right)$$
$$B = \frac{(1 - \gamma\theta)(1 - \lambda)\sigma_z^2 + \lambda\sigma_\epsilon^2}{(1 - \lambda)^2(1 + \theta\varphi)\sigma_z^2 + \lambda^2\sigma_\epsilon^2}$$

Note that the pass through of z_t to $y_{j,t}$ is mitigated by φ (effect of higher wages with linear disutility of labor).

(b) Substitute $y_{j,t}$ in aggregate price index (102), and equate coefficients:

$$\phi_0 = \frac{1}{\varphi + \gamma} \left[\frac{\Omega_s}{2} - \varphi \log \frac{1}{A} + \frac{1}{\theta} \left(\frac{(1 + \varphi\theta)(1 + [\theta - 1]\frac{\lambda}{1-\lambda})^2}{1\theta(\theta - 1)} - \frac{\varphi\theta(\frac{\lambda}{1-\lambda})^2}{2} \right) \sigma_\epsilon^2 \right]$$
$$\sigma_z^2 = \frac{\lambda}{1 - \lambda} \frac{1 - \frac{\lambda}{1-\lambda}}{\theta(\varphi + \gamma)} \sigma_\epsilon^2$$

7.7.2 Non-linear disutility of labor, firm sets price

$$P_{j,t} = \frac{\theta}{\theta - 1} \frac{\mathbb{E}_t[W_t \epsilon_{j,t}^{1-\tau} Y_t | s_{j,t}]}{\mathbb{E}_t[\epsilon_{j,t} Y_t | s_{j,t}]}$$

Replacing N_t with $\int \frac{Y_{j,t}}{\epsilon_{j,t}^{\tau}} dj = P_t^{\theta} Y_t \int P_{j,t}^{-\theta} \epsilon_{j,t}^{1-\tau} dj$

$$P_{j,t} = \frac{\theta}{\theta - 1} \frac{\mathbb{E}_t[P_t^{1+\theta\varphi} \epsilon_{j,t}^{1-\tau} Z_t^{1+\gamma+\varphi} \left(\int P_{j,t}^{-\theta} \epsilon_{j,t}^{1-\tau} dj\right)^{\varphi} | s_{j,t}]}{\mathbb{E}_t[\epsilon_{j,t} Y_t | s_{j,t}]} \tag{90}$$

Key steps:

(a) Substitute guess for $p_{j,t}$ on RHS of (90) and simplify. Equating coefficients with guess,

$$\bar{\mu} = \frac{-\tau\lambda\sigma_{\epsilon}^2 + (\gamma + \varphi + B)(1 - \lambda)\sigma_z^2}{\lambda^2\sigma_{\epsilon}^2 + (1 - \lambda)^2\sigma_z^2}$$

In equilibrium, $B = \bar{\mu}(1 - \lambda)$, which implies:

$$\sigma_z^2 = \frac{\lambda}{1 - \lambda} \frac{\tau + B\frac{\lambda}{1-\lambda}}{\gamma + \varphi} \sigma_{\epsilon}^2$$

7.7.3 Alternative preferences in the model with wages set one period in advance

Consider a more general utility function for households that is non-linear in labor supply,

Households choose the nominal wage for their labor type ($W_{i,t}$) to maximize utility

$$\max_{W_{i,t}} \mathbb{E}_{t-1} \left[\frac{C_{i,t}^{1-\gamma}}{1 - \gamma} - \frac{N_{i,t}^{1+\varphi}}{1 + \varphi} \right]$$

subject to budget constraint

$$P_t C_{i,t} + Q_t B_{i,t} \leq W_{i,t} N_{i,t} + \Pi_{i,t} + B_{i,t-1}$$

and firms' labor demand

$$N_{i,t} = \frac{W_{i,t}}{W_t}^{\varepsilon_w} N_t$$

Maximizing utility with respect to $W_{i,t}$

$$\frac{\partial U}{\partial W_{i,t}} = \mathbb{E}_{t-1} \left[C_{i,t}^{-\gamma} \frac{\partial C_{i,t}}{\partial W_{i,t}} - N_{i,t}^{\varphi} \frac{\partial N_{i,t}}{\partial W_{i,t}} \right]$$

where

$$\frac{\partial N_{i,t}}{\partial W_{i,t}} = -\varepsilon_w \left(\frac{W_{i,t}}{W_t}\right)^{-\varepsilon_w - 1} \frac{N_t}{W_t} = -\varepsilon_w \frac{N_{i,t}}{W_{i,t}}$$

follows from the labor demand schedule of firms and

$$\frac{\partial C_{i,t}}{\partial W_{i,t}} = \frac{1}{P_t}\left(W_{i,t}\frac{N_{i,t}}{W_{i,t}} + N_{i,t}\right) = \frac{N_{i,t}}{P_t}(1-\varepsilon_w)$$

follows from the household's budget constraint and the labor demand schedule of firms. Substituting $\frac{\partial C_{i,t}}{\partial W_{i,t}}$ and $\frac{\partial N_{i,t}}{\partial W_{i,t}}$ in the first order condition yields the optimal wage chosen by household i at $t-1$:

$$\mathbb{E}_{t-1}\left[C_{i,t}^{-\gamma}\frac{N_{i,t}}{P_t}(1-\varepsilon_w) - N_{i,t}^{\varphi}(-\varepsilon_w \frac{N_{i,t}}{W_{i,t}})\right] = 0$$

$$W_{i,t} = \frac{\varepsilon_w}{\varepsilon_w - 1}\frac{\mathbb{E}_{t-1}(N_{i.t}^{\varphi+1})}{\mathbb{E}_{t-1}\left(\frac{N_{i,t}}{C_{i,t}^{\gamma}P_t}\right)}$$

Since consumption will be the same for all households, the wage set will be the same, and we can remove the i index.

$$W_t = \frac{\varepsilon_w}{\varepsilon_w - 1}\frac{\mathbb{E}_{t-1}(N_t^{\varphi+1})}{\mathbb{E}_{t-1}\left(\frac{N_t}{C_t^{\gamma}P_t}\right)}$$

The optimal nominal wage chosen by monopolistically competitive households at $t-1$ will be a markup $(\frac{\varepsilon_w}{\varepsilon_w - 1})$ over the marginal rate of substitution multiplied by the aggregate price level.

Letting $\mu \equiv \frac{\varepsilon_w}{\varepsilon_w - 1}$ and rearranging terms, the households' wage setting equation is

$$1 = \mu\frac{\mathbb{E}_{t-1}(N_t^{\varphi+1})}{\mathbb{E}_{t-1}\left(N_t C_t^{-\gamma}\frac{W_t}{P_t}\right)} \tag{91}$$

Log-linearizing (91) around the steady state (see text in grey):

$$\mathbb{E}_{t-1}\hat{rw}_t = \mathbb{E}_{t-1}(\varphi\hat{n}_t + \gamma\hat{c}_t)$$

With an *iid* shock to sentiments, $\mathbb{E}_{t-1}\hat{n}_t = 0$, since $\hat{y}_t = \hat{n}_t$ by the production function ($Y_t = AN_t$), and $\hat{y}_t = \hat{c}_t$ by the market clearing condition. Then, as in the case with a utility function that is linear in labor supply, we still get a negative relationship between the real wage and inflation:

$$\hat{rw}_t = -\hat{\pi}_t$$

7.8 Fundamental equilibrium with general utility function and general production function

In the fundamental equilibrium, aggregate output and aggregate price index are constant, while the output levels and prices chosen by individual firms responds to idiosyncratic demand ($\epsilon_{j,t}$).

General utility function for households

$$U = \frac{C_t^{1-\gamma}}{1-\gamma} + \Psi(1 - N_t)$$

The resulting optimal labor supply decision of households,

$$\frac{W_t}{P_t} = \Psi Z_t^{\gamma} \tag{92}$$

General production function,

$$Y_{j,t} = \epsilon_{j,t}^{\tau} N_{j,t} \tag{93}$$

Regardless of whether firms choose quantity or price, they are also constrained by demand for their product,

$$Y_{j,t} = \left(\frac{P_t}{P_{j,t}}\right)^{\theta} \epsilon_{j,t} Y_t \tag{94}$$

Firms maximize profits, which is total revenue less total cost:

$$\Pi_{j,t} = P_{j,t}Y_{j,t} - W_t N_{j,t}$$

- When firms choose quantity ($Y_{j,t}$), profit maximization when replacing $P_{j,t}$ with demand (94) and $N_{j,t}$ with production (93):

$$\max_{Y_{j,t}} [P_t Y_{j,t}^{-\frac{1}{\theta}} (\epsilon_{j,t} Y_t)^{\frac{1}{\theta}}] Y_{j,t} - W_t [Y_{j,t} \epsilon_{j,t}^{-\tau}]$$

Taking the derivation with respect to $Y_{j,t}$,

$$P_t \left[\frac{\theta - 1}{\theta} Y_{j,t}^{-\frac{1}{\theta}}\right] (\epsilon_{j,t} Y_t)^{\frac{1}{\theta}} - W_t \epsilon_{j,t}^{-\tau} = 0$$

Note, this is equivalent to

$$\frac{\theta - 1}{\theta} P_{j,t} - W_t \epsilon_{j,t}^{-\tau} = 0$$

Rewriting the first order condition in terms of $Y_{j,t}$,

$$Y_{j,t} = \left[\frac{\theta - 1}{\theta} \frac{1}{W_t/P_t} \epsilon_{j,t}^{\tau + \frac{1}{\theta}} Y_t^{\frac{1}{\theta}}\right]^{\theta}$$

Replacing real wage with (92),

$$Y_{j,t} = \left[\frac{\theta - 1}{\theta} \frac{1}{\Psi} \epsilon_{j,t}^{\tau + \frac{1}{\theta}} Y_t^{\frac{1}{\theta} - \gamma}\right]^{\theta}$$

- When firms choose price ($P_{j,t}$), profit maximization when replacing $Y_{j,t}$ with demand (94) and $N_{j,t}$ with production (93):

$$\max_{P_{j,t}} P_{j,t}[P_t^{\theta} P_{j,t}^{-\theta} \epsilon_{j,t} Y_t] - W_t[Y_{j,t} \epsilon_{j,t}^{-\tau}]$$

  Taking the derivation with respect to $P_{j,t}$,

$$P_{j,t} = \frac{\theta}{\theta - 1} W_t \epsilon_{j,t}^{-\tau}$$

  Replacing W_t with (92),

$$P_{j,t} = \frac{\theta}{\theta - 1} \Psi P_t Y_t^{\gamma} \epsilon_{j,t}^{-\tau}$$

  Of course, this can be translated into the first order condition of the quantity setting problem using the demand curve.

The price setting section uses a household utility function and production function that nests the ones in the quantity setting section. In the latter, $\tau = 0$, which measures the degree to which idiosyncratic demand ($\epsilon_{j,t}$) affects productivity, and $\gamma = 1$, which measures the wealth/substitution effect of household income. If γ increases, the wealth effect dominates, and households require a higher real wage for the same level of aggregate demand.

The firms' first order condition in either case equates marginal benefit with marginal cost, the latter depending on the real wage and hence the household's belief about aggregate output. When monetary policy is incorporated, the central bank's response can affect real wage through the price level.

The fundamental equilibrium in the quantity setting and price setting cases are the same, and characterized by the following equations:

$$Y_t = \left[\int \epsilon_{j,t}^{\frac{1}{\theta}} Y_{j,t}^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}}$$

$$Y_{j,t} = \left[\frac{\theta - 1}{\theta} \frac{1}{\Psi} \epsilon_{j,t}^{\tau + \frac{1}{\theta}} Y_t^{\frac{1}{\theta} - \gamma} dj\right]^{\theta}$$

$$P_t = \frac{W_t}{\Psi Y_t^{\gamma}}$$

$$P_{j,t} = \frac{\theta}{\theta - 1} \epsilon_{j,t}^{-\tau} W_t$$

Note that if Y_t is constant, so is P_t.

7.8.1 Fundamental equilibrium

Substituting $Y_{j,t}$ into the expression for Y_t and using the properties of a moment generating function for a normal random variable, can show that aggregate output is constant in the fundamental equilibrium:

$$\phi_0 = y_t = \frac{[1 + \tau(\theta - 1)]^2}{2\gamma(\theta - 1)} \sigma_{\varepsilon}^2$$

Note that if $\tau = 0$ and $\gamma = 1$, $\phi_0 = \frac{1}{2(\theta-1)}\sigma_\varepsilon^2$, the expression in (112).
When the same general household utility function and firm production function is assumed, note that the steady state of the fundamental equilibrium when firms choose quantity is the same as that of the fundamental equilibrium when firms choose prices (22).

7.8.2 Sentiment equilibrium

$$\sigma_z^2 = \frac{\lambda}{1-\lambda} \frac{(\tau\theta + 1) - \frac{\lambda}{1-\lambda}}{\gamma\theta} \sigma_\varepsilon^2 \tag{95}$$

WIth $\tau = 0$ and $\gamma = 1$,

$$\sigma_z^2 = \frac{\lambda(1-\lambda) - \lambda^2}{(1-\lambda)^2} \sigma_\varepsilon^2$$

7.9 Benchmark Model with Wage Setting Firms

Information structure: In our baseline information structure, we assume that worker-producers have full information while consumers do not. Instead, consumers must infer local conditions from their observation of the local price, Pi. The motivation for this modeling choice is the fact that, in actual economies, consumers generally devote fewer resources to information acquisition than do producers, who must gather information professionally to plan production and trade on the input market. Nevertheless, this asymmetry in information need not be considered a primitive of the economy. In particular, we can imagine that both worker-producers and consumers enter their markets with dispersed priors of equal precision about the state of the economy. Since worker-producers trade on both a local market and a global market, they will perfectly discern local and global conditions. Since consumers interact only in their local market, however, their market experiences will not allow them to do the same.
In the benchmark model (Benhabib et al. (2015)), there is a representative household and a continuum of monopolistic intermediate goods producers indexed by $j \in [0, 1]$. Households supply labor and form *demand schedules* for differentiated goods conditional on shocks that have not yet been realized. The key friction is that intermediate goods firms commit to labor demand and output, based on an imperfect signal of the aggregate demand and firm level demand, prior to goods being produced and exchanged and before marketing clearing prices are realized.

After production decisions are made, the goods market opens, demand is realized, and prices adjust to clear the market. The firms' signal extraction problem can lead to multiple equilibria and endogenous fluctuations in aggregate output.

7.10 Households

The representative household chooses labor N_t to maximize utility

$$\max_{N_t} \log C_t + \Psi(1 - N_t)$$

subject to budget constraint

$$C_t \leq \frac{W_t}{P_t} N_t + \frac{\Pi_t}{P_t}$$

where C_t is aggregate an consumption index, $\frac{W_t}{P_t}$ is the real wage, $\frac{\Pi_t}{P_t}$ is real profit income from all firms, Ψ is disutility of labor. Their first order condition is

$$C_t = \frac{1}{\Psi} \frac{W_t}{P_t} \tag{96}$$

where

$$C_t = \left[\int_0^1 \epsilon_{j,t}^{\frac{1}{\theta}} C_{j,t}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \tag{97}$$

C_t represents an aggregate consumption index, $\theta > 1$ is the elasticity of substitution between goods, $C_{j,t}$ denotes the quantity of good j consumed by the household in period t. The idiosyncratic preference shock for good j is log normally distributed ($\varepsilon_{j,t} \equiv \log \epsilon_{j,t} \sim N(0, \sigma_\varepsilon^2)$. The exponent $\frac{1}{\theta}$ on $\epsilon_{j,t}$ is solely intended to simplify expressions. The household allocates consumption among j goods to maximize C_t for any given level of expenditures $\int_0^1 P_{j,t} C_{j,t} dj$, where $P_{j,t}$ is the price of intermediate good j (see appendix (7.3)).

From optimizing its consumption allocation, household's demand for good j is given by

$$C_{j,t} = \left(\frac{P_t}{P_{j,t}} \right)^{\theta} C_t \epsilon_{j,t} \tag{98}$$

The resulting aggregate price level is obtained by substituting (98) into (97):

$$P_t = \left(\int_0^1 \epsilon_{j,t} P_{j,t} dj \right)^{\frac{1}{1-\theta}}$$

In this model, households form demand schedules for each differentiated good and supply labor, all contingent on shocks to idiosyncratic demand and shocks to aggregate income/consumption are be realized. Let Z_t represent the household's beliefs about aggregate income/consumption at the beginning of period t. Households form consumption *plans* using (98)

$$C_{j,t}(Z_t, \epsilon_{j,t}) = \left(\frac{P_t(Z_t)}{P_{j,t}(Z_t, \epsilon_{j,t})} \right)^{\theta} C_t(Z_t) \epsilon_{j,t} \tag{99}$$

and decide labor supply, using (96) to obtain an implicit function of labor supply as a function of sentiments, $N_t = N(Z_t)$, given a nominal wage W_t

$$P_t(Z_t) = \frac{W_t}{\Psi \left[\frac{1}{P_t(Z_t)} N_t + \frac{\Pi_t(Z_t)}{P_t(Z_t)} \right]} \tag{100}$$

Note that $\Pi_t(Z_t) = P_t(Z_t) Z_t - W_t N_t$.

Intermediate goods firms

The intermediate goods firms decide production level $Y_{j,t}$ without perfect knowledge of idiosyncratic demand or aggregate demand. Instead, they infer $\epsilon_{j,t}$ and $Y_{j,t}$ from a signal $s_{j,t}$ that may be interpreted as early orders, advance sales, or market research:

$$s_{j,t} = \lambda\varepsilon_{j,t} + (1-\lambda)y_t$$

where $\varepsilon_{j,t} \equiv \log\epsilon_{j,t} \sim N(0,\sigma_\varepsilon^2)$ and $y_t \equiv (\log Y_t) - \phi_0 \sim N(0,\sigma_y^2)$. Given the nominal wage, intermediate goods producers choose $Y_{j,t}$ to maximize nominal profits ($\Pi_{j,t} = P_{j,t}Y_{j,t} - W_tN_{j,t}$) subject to production function ($Y_{j,t} = AN_{j,t}$) and demand for its good (98). Substituting out labor demand of firm j ($N_{j,t} = \frac{Y_{j,t}}{A}$) and the price of its good ($P_{j,t}$) using (98), firm j's problem is

$$\max_{Y_{j,t}} \mathbb{E}_t\left[P_tY_{j,t}^{1-\frac{1}{\theta}}(\epsilon_{j,t}Y_t)^{\frac{1}{\theta}} - \frac{W_t}{A}Y_{j,t}|s_{j,t}\right]$$

The first order condition of intermediate goods firm j is given by:

$$\left(1-\frac{1}{\theta}\right)Y_{j,t}^{-\frac{1}{\theta}}\mathbb{E}_t\left[P_t(\epsilon_{j,t}Y_t)^{\frac{1}{\theta}}|s_{j,t}\right] = \frac{W_t}{A}$$

Rearranging terms,

$$Y_{j,t} = \left[\left(1-\frac{1}{\theta}\right)A\mathbb{E}_t\left[(\epsilon_{j,t}Y_t)^{\frac{1}{\theta}}\frac{P_t}{W_t}|s_{j,t}\right]\right]^\theta \tag{101}$$

Substituting P_t with the household's first order condition, $P_t = \frac{1}{\Psi}\frac{W_t}{Y_t}$, where $Y_t = C_t$ due to the absence of savings in this model. As nominal variables are indeterminate in this model, the nominal wage can be normalized to 1.

$$Y_{j,t} = \left[\left(1-\frac{1}{\theta}\right)\frac{A}{\Psi}\mathbb{E}_t[\epsilon_{j,t}^{\frac{1}{\theta}}Y_t^{\frac{1}{\theta}-1}|s_{j,t}]\right]^\theta$$

Higher aggregate output affects firm j's optimal production decision in two ways. On the one hand, higher aggregate output implies aggregate demand will be higher and hence, demand for good j will increase as well, which increases the optimal output level of firm j. On the other hand, given the household's optimal labor supply condition, higher aggregate output implies that the real wage will be higher. Given the normalization of the nominal wage, the aggregate price level will be lower. This will result in a fall in demand for $C_{j,t}$, which decreases firm j's optimal output level. As $\frac{1}{\theta}-1<0$, the latter effect dominates, with the result that firm j's optimal output decreases with aggregate output. Although firms' actions are strategic substitutes, the rational expectations equilibrium is not unique due to information frictions.

7.11 Timing

The timing of this model is as follows: Let Z_t denote aggregate demand and $\epsilon_{j,t}$ represent idiosyncratic preference for good j:

i. Households form labor supply schedule ($N_t(Z_t)$) and demand schedules for each good j, ($C_{j,t}(Z_t, \epsilon_{j,t})$), contingent on shocks to be realized
ii. Z_t, $\epsilon_{j,t}$ realized
iii. Firms receive a private signal of aggregate demand and idiosyncratic preference for their good ($s_{j,t} = \lambda \log \epsilon_{j,t} + (1-\lambda)\log Z_t$)
iv. Firms can not write contingent schedules for their labor demand, otherwise this would remove the possibility of sentiment-driven fluctuations. Instead, firms must commit to production and hence labor demand, based on an imperfect private signal. They produce $Y_{j,t}(s_{j,t})$ and demand labor $N_{j,t}(s_{j,t}) = \frac{Y_{j,t}(s_{j,t})}{A}$.
v. Goods market opens. Z_t, $\epsilon_{j,t}$ observed by everyone. $P_{j,t}$ adjusts so that goods market clears ($C_{j,t} = Y_{j,t}$, $C_t = Y_t$), and $P_t = \frac{1}{\Psi Z_t}$

7.12 Equilibrium

In equilibrium, aggregate output, intermediate goods supply, and the private signal are given by:

$$Y_t = \left[\int \epsilon_{j,t}^{\frac{1}{\theta}} Y_{j,t}^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} \tag{102}$$

$$Y_{j,t} = \left[\left(1 - \frac{1}{\theta}\right)\frac{A}{\Psi}\mathbb{E}[\epsilon_{j,t}^{\frac{1}{\theta}} Y_t^{\frac{1}{\theta}-1} | s_{j,t}]\right]^{\theta} \tag{103}$$

$$s_{j,t} = \lambda \log \epsilon_{j,t} + (1-\lambda)\log Y_t \tag{104}$$

The first equation indicates that in equilibrium, goods markets clear: $Y_t = C_t$, $C_{j,t} = Y_{j,t}$. In the sentiment driven equilibrium, one additional condition applies: that beliefs about aggregate demand are correct in equilibrium.

$$Z_t = Y_t \tag{105}$$

After the realization of Y_t, and after goods markets clear, the aggregate price index, market clearing prices for each good, aggregate labor, and aggregate profits are given by:

$$P_t = \frac{1}{\Psi Y_t} \tag{106}$$

$$P_{j,t} = (\epsilon_{j,t} Y_t)^{\frac{1}{\theta}} Y_{j,t}^{-\frac{1}{\theta}} P_t \tag{107}$$

$$N_t = \int_0^1 N_{j,t} dj = \int_0^1 \frac{Y_{j,t}}{A} dj \tag{108}$$

$$\Pi_t = P_t Y_t - N_t = \frac{1}{\Psi} - N_t \tag{109}$$

In the first equation, actual aggregate price level in equilibrium is determined by realized aggregate output. The second equation indicates that in equilibrium, the market clearing price for good j is determined by realized aggregate output, production of good j, and the realized aggregate price level. In the third equation, labor supply equals aggregate labor demand. In the fourth equation, aggregate profits equal aggregate revenue minus aggregate production costs.

Definition 2. A rational expectations equilibrium is a sequence of allocations $\{C(Z_t), Y(Z_t), C_j(Z_t, \epsilon_{j,t}), Y_j(Z_t, \epsilon_{j,t}), N(Z_t), N_j(Z_t, \epsilon_{j,t}), \Pi(Z_t)\}$, prices $\{P(Z_t), P_j(Z_t, \epsilon_{j,t}), W_t = 1\}$, and a distribution of Z_t, $\mathbf{F}(Z_t)$ such that for each realization of Z_t, (i) equations (99) and (100) maximize household utility given the equilibrium prices $P_t = P(Z_t), P_{j,t} = P_j(Z_t, \epsilon_{j,t})$, and $W_t = 1$ (ii) equation (103) maximizes intermediate goods firm's *expected* profits for all j given the equilibrium prices $P(Z_t), W_t = 1$, and the signal (104) (iii) all markets clear: $C_{j,t} = Y_{j,t}, N(Z_t) = \int N_{j,t}dj$, and (iv) expectations are rational such that the household's beliefs about P_t and Π_t are consistent with its belief about aggregate demand Z_t (according to its optimal labor supply condition) and $Y_t = Z_t$: actual aggregate output follows a distribution consistent with $\mathbf{F}$.

There exist two rational expectations equilibria: (1) A fundamental equilibrium with a degenerate distribution of sentiments, where aggregate output and prices are all constant and where sentiments play no role in determining the level of aggregate output (2) A stochastic equilibrium where sentiments matter and the volatility of beliefs about aggregate demand is endogenously determined and equal to the variance of aggregate output.

7.12.1 Fundamental equilibrium

Under perfect information, firms receive signals that reveal their idiosyncratic demand shocks, and we will show that there is a unique rational expectations equilibrium in which output, aggregate demand, and the aggregate price level are constant. Using the equilibrium conditions in (103), (102), (107), and (106), $Y_t, P_t, Y_{j,t}$ and $P_{j,t}$ in the fundamental equilibrium are as follows: From (103)

$$Y_{j,t} = \left[\left(1 - \frac{1}{\theta}\right)\frac{A}{\Psi}\epsilon_{j,t}^{\frac{1}{\theta}} Y_t^{\frac{1}{\theta}-1}\right]^{\theta} \tag{110}$$

Using (102), and substituting $Y_{j,t}$ with (110)

$$\begin{aligned} Y_t &= \left[\int_0^1 \epsilon_{j,t}^{\frac{1}{\theta}} Y_{j,t}^{1-\frac{1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} \\ &= \left[\int_0^1 \epsilon_{j,t}^{\frac{1}{\theta}}\left[\left(1 - \frac{1}{\theta}\right)\frac{A}{\Psi}\epsilon_{j,t}^{\frac{1}{\theta}} Y_t^{\frac{1}{\theta}-1}\right]^{\theta-1} dj\right]^{\frac{\theta}{\theta-1}} \\ &= \left(1 - \frac{1}{\theta}\right)\frac{A}{\Psi}\left[\int_0^1 \epsilon_{j,t} dj\right]^{\frac{1}{\theta-1}} \end{aligned}$$

Let variables with * denote their counterparts in the fundamental equilibrium. As $C_t = Y_t$ in equilibrium,

$$C^* = Y^* = \left(1 - \frac{1}{\theta}\right)\frac{A}{\Psi}\left[\int_0^1 \epsilon_{j,t} dj\right]^{\frac{1}{\theta-1}} \tag{111}$$

Using (106), the equilibrium aggregate price level is

$$P^* = \frac{1}{\Psi Y^*} = \frac{\theta}{\theta-1}\frac{1}{A}\left[\int_0^1 \epsilon_{j,t} dj\right]^{\frac{1}{1-\theta}}$$

In the fundamental equilibrium, as Y_t is known, $s_{j,t}$ reveals $\epsilon_{j,t}$ perfectly. Any shift in $\epsilon_{j,t}$ results in a corresponding change in $Y_{j,t}$ without affecting $P_{j,t}$. Substituting the previous expressions for Y_t, P_t, and $Y_{j,t}$ into (107)

$$P_{j,t} = \frac{\theta}{\theta-1}\frac{1}{A}$$

Let $y^* \equiv \log(Y^*)$. Without loss of generality, let $\left(1-\frac{1}{\theta}\right)\frac{A}{\Psi} = 1$. Then another way of expressing (111) (see gray text for details):

$$y^* = \frac{1}{2(\theta-1)}\sigma_\varepsilon^2 \tag{112}$$

(113)

Aggregate production in the fundamental equilibrium is given by (111):

$$Y^* = \left(1-\frac{1}{\theta}\right)\frac{A}{\Psi}\left[\int_0^1 \epsilon_{j,t} dj\right]^{\frac{1}{\theta-1}}$$

Without loss of generality, let $\left(1-\frac{1}{\theta}\right)\frac{A}{\Psi} = 1$. Then

$$\begin{aligned} Y^* &= \left[\int_0^1 \epsilon_{j,t} dj\right]^{\frac{1}{\theta-1}} \\ y^* &= \log Y^* = \frac{1}{\theta-1}\log \mathbb{E}_t[\epsilon_{j,t}] \\ &= \frac{1}{\theta-1}\log \mathbb{E}_t[e^{\varepsilon_{j,t}}] \end{aligned}$$

where $\varepsilon_{j,t} \equiv \log \epsilon_{j,t}$. As $\varepsilon_{j,t} \sim N(0, \sigma_\varepsilon^2)$, by the properties of its moment generating function:

$$\begin{aligned} \mathbb{E}_t[e^{\varepsilon_{j,t}}] &= e^{\mathbb{E}_t(\varepsilon_{j,t})+\frac{1}{2}\mathrm{Var}_t(\varepsilon_{j,t})} \\ &= e^{\frac{\sigma_\varepsilon^2}{2}} \end{aligned}$$

Then it follows that

$$y^* = \frac{1}{2(\theta-1)}\sigma_\varepsilon^2$$

7.12.2 Sentiment-driven equilibrium

When firms face information frictions, there exists a sentiment driven equilibrium, in addition to the fundamental equilibrium. The sentiment driven equilibrium is a rational expectations equilibrium where aggregate output is not constant but equal to a sentiment (Z_t). Let $\hat{z}_t$ and $\hat{y}_t$ denote Z_t and Y_t in log deviation from the steady state of this equilibrium, respectively.[13] $\hat{z}_t \sim N(0, \sigma_z^2)$, where σ_z^2 is a constant to be determined below.

Equation (103) gives firm j's optimal output conditional on its signal. As it is derived using equations (96) and (98), it already incorporates market clearing for labor and consumption.

$$Y_{j,t} = \left[\left(1 - \frac{1}{\theta}\right)\frac{A}{\Psi}\mathbb{E}[\epsilon_{j,t}^{\frac{1}{\theta}} Y_t^{\frac{1}{\theta}-1} | s_{j,t}]\right]^{\theta}$$

Firm j's private signal is

$$s_{j,t} = \lambda\hat{\varepsilon}_{j,t} + (1-\lambda)\hat{z}_t$$

Log-linearizing around the steady state: (see text in grey)

$$\hat{y}_{j,t} = \mathbb{E}_t[\hat{\varepsilon}_{j,t} + (1-\theta)\hat{y}_t | s_{j,t}]$$

Details: Let $y_{j,t} \equiv \log Y_{j,t}$, $\varepsilon_{j,t} \equiv \log \epsilon_{j,t}$, $\gamma \equiv \left(1 - \frac{1}{\theta}\right)\frac{A}{\Psi}$, $y_t \equiv \log Y_t$. Then the firms' first order condition:

$$e^{y_{j,t}} = \left[\gamma \mathbb{E}_t\left(e^{\frac{1}{\theta}\varepsilon_{j,t}} e^{\frac{1-\theta}{\theta}y_t} | s_{j,t}\right)\right]^{\theta}$$

Let $\phi_t \equiv \gamma \mathbb{E}_t\left(e^{\frac{1}{\theta}\varepsilon_{j,t}} e^{\frac{1-\theta}{\theta}y_t} | s_{j,t}\right)$. Then taking the Taylor expansion around the steady state:

$$ss + e^{y_j}\hat{y}_{j,t} = ss + \theta\phi^{\theta-1}\underbrace{\gamma e^{\frac{1}{\theta}\varepsilon_j + \frac{1-\theta}{\theta}y}}_{\phi}\left(\frac{1}{\theta}\hat{\varepsilon}_{j,t} + \frac{1-\theta}{\theta}\hat{y}_t\right)$$

$$e^{y_j}\hat{y}_{j,t} = \theta\phi^{\theta}\left(\frac{1}{\theta}\hat{\varepsilon}_{j,t} + \frac{1-\theta}{\theta}\hat{y}_t\right)$$
$$= \phi^{\theta}\hat{\varepsilon}_{j,t} + \phi^{\theta}(1-\theta)\hat{y}_t$$

Since $e^{y_j} = \phi^{\theta}$,

$$\hat{y}_{j,t} = \mathbb{E}_t[\hat{\varepsilon}_{j,t} + (1-\theta)\hat{y}_t | s_{j,t}]$$

■

[13] See appendix (7.4) for a calculation of the steady state in this equilibrium.

Let $\mu = \arg\min_\mu \mathbb{E}[D_\varepsilon \varepsilon_{j,t} + D_z z_t - \mu s_{j,t}]^2$. Then

$$\mu = \frac{\text{cov}(D_\varepsilon \varepsilon_{j,t} + D_z z_t, s_{j,t})}{\text{var}(s_{j,t})}$$

The conditional expectation is the best forecast under a squared loss function. (From C&G) Justification for the inference problem and the REE: Key feature of the signal extraction problem is that the precision of the signal (informativeness of idiosyncratic productivity) depends on the nature of average actions across the population, and therefore, on the average reaction of other consumers to their own price signals. A REE is a situation in which the individual reaction to the signal is consistent with its actual precision (ie is an optimal response to the average reaction of others). This is justification for Bayesian weighting using σ_z^2. Since we assume that all stochastic elements are normal, the optimal forecasting strategy is linear.

Conditional on its signal, firm j's best response is

$$\begin{aligned}\hat{y}_{j,t} &= \frac{\lambda\sigma_\varepsilon^2 + (1-\theta)(1-\lambda)\sigma_z^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2} s_{j,t} \\ &= \frac{\lambda\sigma_\varepsilon^2 + (1-\theta)(1-\lambda)\sigma_z^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2}(\lambda\hat{\varepsilon}_{j,t} + (1-\lambda)\hat{z}_t)\end{aligned}$$

Aggregate supply is then

$$\begin{aligned}\hat{y}_t &= \int_0^1 \hat{y}_{j,t} dj \\ &= \frac{\lambda\sigma_\varepsilon^2 + (1-\theta)(1-\lambda)\sigma_z^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2}(1-\lambda)\hat{z}_t\end{aligned}$$

In equilibrium, household's beliefs about aggregate demand are correct ($\hat{y}_t = \hat{z}_t$). This implies

$$1 = \frac{\lambda\sigma_\varepsilon^2 + (1-\theta)(1-\lambda)\sigma_z^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2}(1-\lambda)$$

Then, the volatility of actual aggregate output and beliefs about aggregate demand are determined by the parameters of the model. If $\lambda \in (0, \frac{1}{2})$ and $\sigma_\varepsilon^2 > 0$, then there exists a sentiment driven rational expectations equilibrium with $\hat{y}_t = \hat{z}_t$ where

$$\sigma_y^2 = \sigma_z^2 = \underbrace{\frac{\lambda(1-2\lambda)}{(1-\lambda)^2\theta}\sigma_\varepsilon^2}_{B} \tag{114}$$

Let B denote the volatility of sentiments under the baseline model. The volatility of the sentiment shock must be commensurate with the degree of complementarity/substitutability in actions across firms (θ), information content of the private signal (λ), and the volatility of idiosyncratic demand (σ_ε^2), all of which affect the firm's response to a sentiment shock.

Note that if $\lambda = 1$, the signal contains only the idiosyncratic preference shock, the result is that an equilibrium with constant output is the unique equilibrium. If $\lambda = 0$ or $\sigma_\varepsilon^2 = 0$, then the private signal conveys only aggregate components. The result is also that the unique equilibrium is the fundamental equilibrium, due to substitutability of firms' outputs.

The intuition for why the sentiment-driven equilibrium is a rational expectations equilibrium is as follows: Given the parameters of the model, σ_z^2 is determined such that for any aggregate demand sentiment, all firms misattribute enough of the sentiment component of their signal to an idiosyncratic preference shock such that aggregate output will be equal to the sentiment in equilibrium. The volatility of the sentiment process (σ_z^2) determines how much firms attribute their signal to $\hat{z}_t$. In particular, when firms' actions are strategic substitutes, the optimal output of a firm is declining in σ_z^2 as this leads the firms to attribute more of the signal to an aggregate demand shock. Since firms' optimal output depends negatively on the level of $\hat{z}_t$ and positively on the idiosyncratic preference shock $\hat{\varepsilon}_{j,t}$, if they are unable to distinguish between the two components in their signal, then there can be a coordinated over-production (under-production) in response to a positive (negative) aggregate sentiment shock, such that $\hat{y}_t$ equals $\hat{z}_t$ in equilibrium if σ_z^2 is as in (114). The rational expectations equilibrium pins down the variance of the sentiment distribution, although sentiments are extrinsic. The result is an additional rational expectations equilibrium that is characterized by aggregate fluctuations in output and employment despite the lack of fundamental aggregate shocks.

7.13 Monetary Policy with Sticky Wages (Calvo)

By introducing sticky wages, the mechanism through which a sentiment shock is fulfilled changes, which has implications for the equilibrium real wage. The response of a central bank to inflation can affect the degree of substitutability or complementarity in the firm's production decision.

In the **benchmark model**, nominal variables are indeterminate. Normalizing w_t does not link the real wage to inflation, and the equilibrium is achieved through an intra-temporal effect. The mechanism through which a sentiment shock is realized is as follows: For the positive sentiment shock to be self-fulfilling, the real wage must increase (which implies that the price level falls, considering the normalized nominal wage). As the price level falls, households increase consumption and supply more labor, resulting in a self-fulfilling equilibrium. The firm's optimal response to sentiments is determined by the effects of this mechanism on the real wage. A positive sentiment shock is fulfilled by an increase in the real wage. As a result, firms want to decrease production in response to sentiments.

In the alternative model with **sticky wages, a la Calvo** and a central bank that targets wage inflation, the real wage is now a function of price inflation.

The mechanism through which a positive sentiment shock is realized is now intertemporal. With wage stickiness, households no longer supply labor optimally at a given real wage. Instead, a proportion $1-\theta_w$ of households can they set wages optimally in the previous period, given information available, and supply labor elastically. For a positive sentiment to be self-fulfilling now, the real interest rate must fall, inducing households to shift consumption to the current period. the real interest rate decreases in one of two ways, either through a decrease in the nominal interest rate (which occurs if there is a decrease in wage inflation), or an increase in expected price inflation. The fact that deflation is necessary for a positive sentiment to be realized implies that the real wage must increase, preserving the substitutability in firms' production. In equilibrium, the price level is still negatively related to sentiments but for a different reason: it is because inflation must fall so that the interest rate responds to fulfill a positive sentiment.

7.14 Households

7.14.1 Optimal Wage Setting

Each household specializes in a different type of labor, which it supplies monopolistically. Alternatively, one can consider a continuum of unions, each of which represents a set of households specialized in a type of labor, and sets the wage on their behalf. At $t-1$, let households indexed by $i \in [0,1]$, offering labor type $N_{i,t}$, choose wage $W_{i,t}$ to maximize utility:

$$\max_{W_{i,t}} \mathbb{E}_t[\log C_{i,t} + \Psi(1 - N_{i,t})]$$

subject to the budget constraint

$$P_t C_{i,t} \leq W_{i,t} N_{i,t} + \Pi_{i,t} \tag{115}$$

and subject to the labor demand schedule of firms for labor type i, which is obtained from the labor expenditure minimization by firms[14]

[14]Firm j produces output $Y_{j,t}$ according to the production function

$$Y_{j,t} = AN_{j,t}$$

$N_{j,t}$ is an index of labor input used by firm j and is defined as

$$N_{j,t} = \left[\int_0^1 N_{i,j,t}^{1-\frac{1}{\theta_w}}\, di\right]^{\frac{\theta_w}{\theta_w - 1}}$$

$N_{i,j,t}$ is the quantity of type i labor employed by firm j in period t. The parameter θ_w represents the elasticity of substitution among labor varieties. From firm minimization of labor expenditure, the following labor demand schedules are obtained:

$$N_{i,j,t} = \left(\frac{W_{i,t}}{W_t}\right)^{-\theta_w} N_{j,t}$$

W_t is the aggregate nominal wage index, defined as

$$W_t \equiv \left[\int_0^1 W_{i,t}^{1-\theta_w}\, di\right]^{\frac{1}{1-\theta_w}}$$

$$N_{i,t} = \left(\frac{W_{i,t}}{W_t}\right)^{-\theta_w} N_t \tag{116}$$

Household i's consumption index is given by

$$C_{i,t} = \left[\int_0^1 \epsilon_{i,j,t}^{\frac{1}{\theta}} C_{i,j,t}^{1-\frac{1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}}$$

where $C_{i,j,t}$ represents household i's consumption of good j.

The first order condition of household i:

$$\frac{\partial U}{\partial W_{i,t}} = \mathbb{E}_t\left[\frac{1}{C_{i,t}}\frac{\partial C_{i,t}}{\partial W_{i,t}} - \Psi\frac{\partial N_{i,t}}{\partial W_{i,t}}\right]$$

where

$$\frac{\partial N_{i,t}}{\partial W_{i,t}} = -\theta_w\left(\frac{W_{i,t}}{W_t}\right)^{-\theta_w-1}\frac{N_t}{W_t} = -\theta_w\frac{N_{i,t}}{W_{i,t}}$$

follows from the labor demand schedule of firms (116) and

$$\frac{\partial C_{i,t}}{\partial W_{i,t}} = \frac{1}{P_t}\left(W_{i,t}\frac{\partial N_{i,t}}{\partial W_{it}} + N_{i,t}\right) = \frac{N_{i,t}}{P_t}(1-\theta_w)$$

follows from the household's budget constraints (115). Substituting $\frac{\partial C_{i,t}}{\partial W_{i,t}}$ and $\frac{\partial N_{i,t}}{\partial W_{i,t}}$ in the first order condition yields the optimal wage chosen by household i at t:

$$\mathbb{E}_t\left[\frac{1}{C_{i,t}}\frac{N_{i,t}}{P_t}(1-\theta_w) + \Psi\theta_w\frac{N_{i,t}}{W_{i,t}}\right] = 0$$

$$W_{i,t} = \frac{\theta_w}{\theta_w - 1}\mathbb{E}_t(\Psi C_{i,t} P_t)$$

Since consumption will be the same for all households, the wage set will be the same, and we can remove the i index.

$$W_t = \frac{\theta_w}{\theta_w - 1}\mathbb{E}_t(\Psi C_t P_t)$$

Aggregating across firms, the demand for type i labor is:

$$N_{i,t} = \int_0^1 N_{i,j,t} dj = \left(\frac{W_{i,t}}{W_t}\right)^{-\theta_w}\int_0^1 N_{j,t} dj = \left(\frac{W_{i,t}}{W_t}\right)^{-\theta_w} N_t$$

The optimal nominal wage chosen by monopolistically competitive households at t will be a markup $(\frac{\theta_w}{\theta_w - 1})$ over the marginal rate of substitution multiplied by the aggregate price level.

Letting $\mu \equiv \frac{\theta_w}{\theta_w - 1}$ and rearranging terms, the households' wage setting equation is

$$1 = \mu \Psi \mathbb{E}_t \left(\frac{P_t}{W_t} C_t \right) \tag{117}$$

Log-linearizing (117) around the steady state (see text in grey):

$$0 = \mathbb{E}_t(-\hat{w}_t^r + \hat{c}_t)$$

$$\hat{w}_t^r = \hat{c}_t \tag{118}$$

Details: Let $rw_t \equiv \log \frac{W_t}{P_t}$, $c_t \equiv \log C_t$, and variables with a hat denote log deviation from steady state. Then (118) can be rewritten as:

$$\begin{aligned} 1 &= \mu \Psi \mathbb{E}_t(e^{-rw_t + c_t}) \\ 1 &= ss + \mu \Psi e^{-rw + c} \mathbb{E}_t(-\hat{w}_t^r + \hat{c}_t) \\ 0 &= \mathbb{E}_t(-\hat{w}_t^r + \hat{c}_t) \end{aligned}$$

$$\hat{w}_t^r = -\hat{c}_t$$

■

7.15 Intermediate goods firms

The firms' first order condition (101)

$$Y_{j,t} = \left[\left(1 - \frac{1}{\theta} \right) A \mathbb{E}_t \left(\epsilon_{j,t}^{\frac{1}{\theta}} Y_t^{\frac{1}{\theta}} \frac{P_t}{W_t} | s_{j,t} \right) \right]^{\theta}$$

Log-linearizing around the steady state: (see text in grey)

$$\hat{y}_{j,t} = \mathbb{E}_t[\hat{\varepsilon}_{j,t} + \hat{y}_t - \theta \hat{w}_t^r | s_{j,t}]$$

Details: Let $y_{j,t} \equiv \log Y_{j,t}$, $\varepsilon_{j,t} \equiv \log \epsilon_{j,t}$, $\gamma \equiv \left(1 - \frac{1}{\theta}\right) A$, $rw_t \equiv \log \frac{W_t}{P_t}$. Then the firms' first order condition:

$$e^{y_{j,t}} = \left[\gamma \mathbb{E}_t \left(e^{\frac{1}{\theta} \varepsilon_{j,t}} e^{\frac{1}{\theta} y_t - rw_t} | s_{j,t} \right) \right]^{\theta}$$

Let $\phi_t \equiv \gamma \mathbb{E}_t \left(e^{\frac{1}{\theta}\varepsilon_{j,t}} e^{\frac{1}{\theta}y_t - rw_t} | s_{j,t} \right)$. Then taking the Taylor expansion around the steady state:

$$ss + e^{y_j}\hat{y}_{j,t} = ss + \theta\phi^{\theta-1} \underbrace{\gamma e^{\frac{1}{\theta}\varepsilon_j + \frac{1}{\theta}y - rw}}_{\phi} \left(\frac{1}{\theta}\hat{\varepsilon}_{j,t} + \frac{1}{\theta}\hat{y}_t - \hat{w}^r_t \right)$$

$$e^{y_j}\hat{y}_{j,t} = \theta\phi^{\theta} \left(\frac{1}{\theta}\hat{\varepsilon}_{j,t} + \frac{1}{\theta}\hat{y}_t - \hat{w}^r_t \right)$$

$$= \phi^{\theta}\hat{\varepsilon}_{j,t} + \phi^{\theta}\hat{y}_t - \theta\phi^{\theta}\hat{w}^r_t$$

Since $e^{y_j} = \phi^{\theta}$,

$$\hat{y}_{j,t} = \mathbb{E}_t[\hat{\varepsilon}_{j,t} + \hat{y}_t - \theta\hat{w}^r_t | s_{j,t}]$$

■

7.16 Central bank

Taylor rule targets wage inflation

$$i_t = \rho + \phi^w_\pi \pi^w_t + \phi_y \hat{y}_t$$

7.17 Equilibrium

In equilibrium, when a fraction of households/labor types can reset their nominal wage, wage inflation is as follows:

$$\pi^w_t = \beta\mathbb{E}_t\pi^w_{t+1} - \lambda_w\hat{\mu}^w_t$$

where $\hat{\mu}^w_t \equiv \mu^w_t - \mu^w$ denotes deviations of the wage markup from its steady state level. As the marginal rate of substitution is $\log\psi c_t$, $\hat{\mu}^w_t = \hat{w}^r_t - \hat{c}_t$.
Optimal inter-temporal consumption is given by the Euler equation

$$\hat{c}_t = \mathbb{E}_t\hat{c}_{t+1} - \frac{1}{\sigma}(i_t - \rho - \mathbb{E}_t\hat{\pi}_{t+1}) \tag{119}$$

Firm production, conditional on signal $s_{j,t}$ is

$$\hat{y}_{j,t} = \mathbb{E}_t[\hat{\varepsilon}_{j,t} + \hat{y}_t - \theta\hat{w}^r_t | s_{j,t}]$$

where

$$s_{j,t} = \lambda\hat{\varepsilon}_{j,t} + (1-\lambda)\hat{z}_t$$

The central bank follows the policy rule

$$i_t = \rho + \phi^w_\pi \hat{\pi}^w_t + \phi_y \hat{y}_t$$

As there are no savings in this model, market clearing implies

$$\hat{y}_t = \hat{c}_t$$

The real wage identity can be used to determine price inflation in equilibrium:

$$r\hat{w}_{t+1} = \hat{w}^r_t + \mathbb{E}_t\hat{\pi}^w_{t+1} - \mathbb{E}_t\hat{\pi}_{t+1}$$

Lastly, beliefs about aggregate demand are correct:

$$\hat{z}_t = \hat{y}_t$$

7.18 Effect of an *iid* shock to sentiments

Consider the effect of an *iid* shock to sentiments, z_t, on the volatility of sentiments in equilibrium. To find $\hat{w}^r_t, \hat{\pi}_t, \hat{\pi}^w_t$ in terms of $\hat{z}_t$, guess and verify the following policy functions:

$$\begin{aligned}
\hat{c}_t &= a_c\hat{z}_t + b_c\hat{w}^r_{t-1} \\
\hat{w}^r_t &= a_r\hat{z}_t + b_r\hat{w}^r_{t-1} \\
\pi^w_t &= a_w\hat{z}_t + b_w\hat{w}^r_{t-1} \\
\pi_t &= a_p\hat{z}_t + b_p\hat{w}^r_{t-1}
\end{aligned}$$

The result (see appendix 7.6 for details)

$$\begin{aligned}
\hat{c}_t &= \hat{z}_t && (120) \\
\hat{w}^r_t &= \frac{\phi^w_\pi\lambda_w + (\sigma+\phi_y)}{1+\phi^w_\pi\lambda_w}\hat{z}_t && (121) \\
&= \left(1 - \left[\frac{1-(\sigma+\phi_y)}{\phi^w_\pi\lambda_w + 1}\right]\right)\hat{z}_t && (122) \\
\pi^w_t &= \frac{\lambda_w(1-[\sigma+\phi_y])}{1+\phi^w_\pi\lambda_w}\hat{z}_t && (123) \\
\pi_t &= \left[\frac{(\lambda_w+1)(1-[\sigma+\phi_y])}{\phi^w_\pi\lambda_w + 1} - 1\right]\hat{z}_t + \hat{w}^r_{t-1} && (124)
\end{aligned}$$

The firms' optimal production decision is:

$$\hat{y}_{j,t} = \mathbb{E}_t(\hat{\varepsilon}_{j,t} + \hat{y}_t - \theta\hat{w}^r_t|s_{j,t})$$

Incorporating the relationship between the real wage and sentiments (121), as implied by the household's optimal inter-temporal consumption decision, optimal wage setting under Calvo wage rigidity, and the Taylor rule:

$$\hat{y}_{j,t} = \mathbb{E}_t\left[\hat{\varepsilon}_{j,t} + \left(1 - \theta\underbrace{\left[\frac{\phi^w_\pi\lambda_w + (\sigma+\phi_y)}{1+\phi^w_\pi\lambda_w}\right]}_{A}\right)\hat{z}_t|s_{j,t}\right] \qquad (125)$$

(126)

7.19 Discussion

In the case of sticky wages and a central bank that targets wage inflation, the sequence of steps by which a positive sentiment shock is self-fulfilling:

- As before, with a simpler implementation of wage stickiness, for consumption to increase on the HH side, the real interest rate must fall. The difference is that the real interest rate falls not only when the nominal interest rate decreases, but also when expected price inflation decreases. The latter is no longer zero in response to an *iid* sentiment shock if the central bank targets wage inflation, but is equal to the real wage.
  - For the nominal interest rate to fall, wage inflation must decrease
  - For expected price inflation to increase, either the real wage increases or the current price level falls.

  Note that for $\gamma + \phi_y \geq 0$ and $\phi_\pi^w > 0$, price inflation decreases in response to a sentiment shock ($\frac{\partial \pi_t}{\partial z_t} < 0$):

  $$\frac{\partial \pi_t}{\partial z_t} = \frac{(\lambda_w + 1)(1 - [\gamma + \phi_y])}{\phi_\pi^w \lambda_w + 1} - 1 < 0$$

  Moreover, following a positive sentiment shock, price inflation falls falls by more than wage inflation ($\frac{\partial \pi_t}{\partial z_t} < \frac{\partial \pi_t^w}{\partial z_t}$):

  $$\begin{aligned} \frac{\partial \pi_t}{\partial z_t} &= \frac{\partial \pi_t^w}{\partial z_t} - \left(1 - \frac{1 - [\gamma + \phi_y]}{\phi_\pi^w \lambda_w + 1}\right) \\ &= \frac{\partial \pi_t^w}{\partial z_t} - \underbrace{\left(\frac{\phi_\pi^w \lambda_w + \gamma + \phi_y}{\phi_\pi^w \lambda_w + 1}\right)}_{>0} \end{aligned}$$

  Next, note that $\frac{\partial \pi_t^w}{\partial z_t} < 0$ if $\gamma + \phi_y \geq 1$.[15]
- On the firm side, this implies that as a result of a positive sentiment shock, the real wage increases ($\frac{\partial w_t^r}{\partial z_t}$), raising marginal cost. Then, as before, the optimal response of a firm to a sentiment shock will be to reduce production.
- The key to understanding the role of wage flexibility and the strength of the central bank's response: for the positive sentiment shock to be fulfilled on the HH side, the real interest rate must fall through a combination of a decrease in the nominal interest rate and an increase in expected price inflation (hence fall in current price level). Higher wage flexibility implies that wage inflation will fall by more in response to a positive sentiment shock, thereby mitigating the degree to which the current price level needs to fall. The result is that the real wage does not increase by as much (show). A stronger central bank response to wage inflation caps the fall in wage inflation. As the nominal interest rate will

[15] In the case of $0 \geq \gamma + \phi_y < 1$, a positive sentiment shock leads to an increase in wage inflation. However, it is still the case that price inflation decreases, and as a result, the real wage still increases. Although the nominal interest rate increases in response to wage inflation rising, expected price inflation increases when the price level falls. While the fall in the real interest rate is mitigated by the rise in the nominal interest rate, a self-fulfilling equilibrium is possible since $\gamma < 1$ implies that only a small decrease in the real interest rate is sufficient to increase household consumption.

7.19.1 Role of wage flexibility (λ_w)

This section refers to the policy functions (121-124), and the analysis is for $\gamma + \phi_y > 1$ and $\phi_\pi^w > 1$.

- **Summary**: As the HH's optimal labor supply decision no longer holds (wages are sticky), a positive sentiment is fulfilled through a decrease in the real interest rate, not an increase in the real wage. The real wage does not change in order for a sentiment shock to be fulfilled. Instead, what happens to the real wage is a consequence of how the real interest rate changes in order for a sentiment shock to be fulfilled. The degree of wage flexibility determines how the real interest rate will fall in order to fulfill a positive sentiment shock. λ_w affects the *composition* of changes in wage inflation and price inflation necessary for a fall in the real interest rate to fulfill a sentiment shock. With Calvo wage setting, expected price inflation is no longer zero, but equal to the real wage. Then, the real interest rate can fall in at least two ways, either the nominal interest rate falls or expected price inflation increases (current price level decreases).

$$r_t = i_t - \mathbb{E}_t \pi_{t+1}$$

where

$$\mathbb{E}_t \pi_{t+1} \equiv \mathbb{E}_t p_{t+1} - p_t$$

In this analysis, consider how the real interest rate decreases in order for a positive sentiment shock to be fulfilled, and then the implications for the real wage. If, for example, the real wage increases by less, then firm production is characterized by less substitutability, which implies that the volatility of output is higher in equilibrium. In a self-fulfilling equilibrium, increasing the degree of wage flexibility or the central bank's response to wage inflation results in a composition of changes to wage inflation and price inflation such that the real wage does not increase by as much. When the real wage increase by less, so does marginal cost. The result is less substitutability in firms' production with respect to aggregate production (sentiments) and firms increase production. As aggregate supply will exceed aggregate demand, for markets to clear, sentiment volatility must be higher in order for firms to attribute more of their signal to z_t and reduce output in response. The result is that sentiment volatility must be higher in equilibrium.

- **Alternative discussion with NKPC for wage inflation**

$$\begin{aligned}\pi_t^w &= \beta \mathbb{E}_t \pi_{t+1}^w - \lambda_w \mu_t^w \\ &= \beta \mathbb{E}_t \pi_{t+1}^w - \lambda_w [w_t^r - c_t]\end{aligned}$$

Using $\mathbb{E}_t \pi_{t+1}^w = 0$, replacing w_t^r with the real wage identity, and rearranging terms,

$$\pi_t^w = -\frac{\lambda_w}{1+\lambda_w}(\pi_t + c_t - w_{t-1}^r)$$

Note that $\frac{\partial(\frac{\lambda_w}{1+\lambda_w})}{\partial \lambda_w} > 0$. For a given positive sentiment shock to be fulfilled on the household side, the nominal interest rate must fall following a decrease in wage inflation. However, by the NKPC, price inflation must fall by more than wage inflation when consumption increases. As λ_w increases, the less price inflation needs to fall in the self-fulfilling equilibrium.

- $\lambda_w = 0$ **(completely sticky wages)**: When wages are unadjustable, wage inflation is equal to zero, and the nominal interest rate does not change. Then, the real interest rate falls solely through an increase in expected price inflation (fall in p_t).

$$\begin{aligned} \hat{w}_t^r &= (\gamma + \phi_y)\hat{z}_t \\ \pi_t^w &= 0 \\ \pi_t &= -(\gamma + \phi_y)\hat{z}_t + \hat{w}_{t-1}^r \end{aligned}$$

- $\lambda_w \to \infty$ **(completely flexible wages)**: When wages are flexible, wage inflation decreases (w_t falls) in order for the nominal interest rate to fall. Then, the real interest rate falls through a combination of an increase in expected price inflation (fall in p_t) and a decrease in the nominal interest rate. Therefore, expected price inflation does not need to increase by as much, relative to the case where wages are completely sticky, and so p_t **falls by less**. **Since w_t falls and p_t falls by less, w_t^r increases by less**. As $\lambda_w \to \infty$,

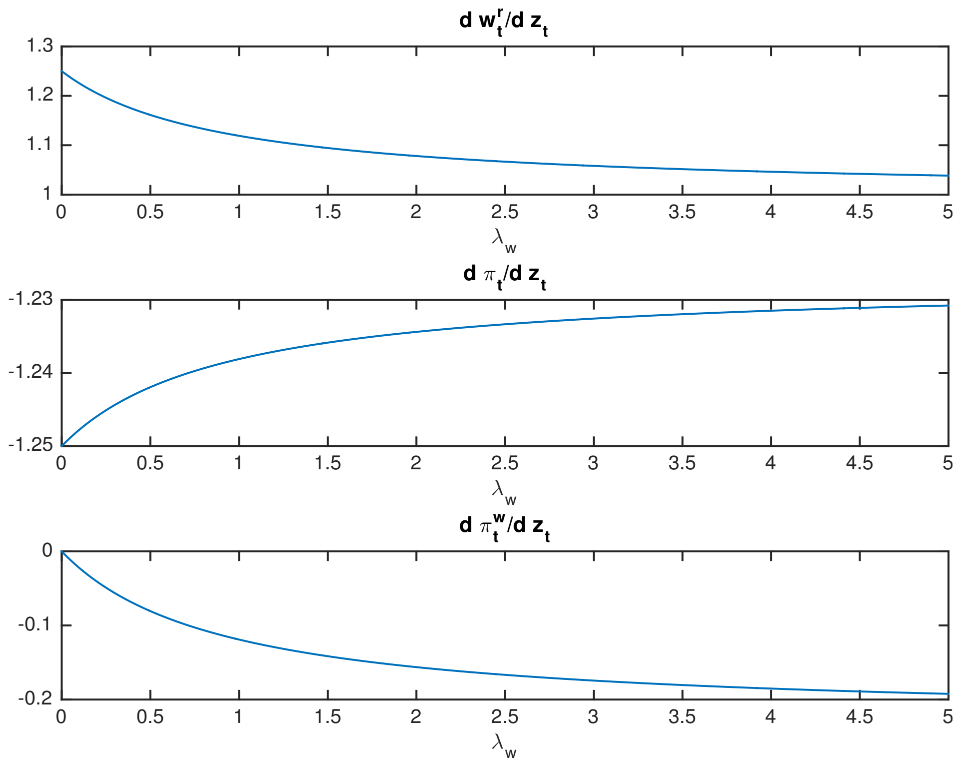
$$\begin{aligned} \hat{w}_t^r &= \frac{\phi_\pi^w + \frac{\gamma+\phi_y}{\lambda_w}}{\frac{1}{\lambda_w} + \phi_\pi^w}\hat{z}_t \to \hat{z}_t \\ \pi_t^w &= \frac{1-(\gamma+\phi_y)}{\frac{1}{\lambda_w} + \phi_\pi^w}\hat{z}_t \to \frac{1-(\gamma+\phi_y)}{\phi_\pi^w}\hat{z}_t \\ \pi_t &= \left[\frac{\left(1+\frac{1}{\lambda_w}\right)[1-(\gamma+\phi_y)]}{\phi_\pi^w + \frac{1}{\lambda_w}} - 1\right]\hat{z}_t + \hat{w}_{t-1}^r \to \left[\frac{1-(\gamma+\phi_y)}{\phi_\pi^w} - 1\right]\hat{z}_t + \hat{w}_{t-1}^r \end{aligned}$$

Note that under perfectly flexible wages, the central bank's response to wage inflation (ϕ_π^w) has no effect on the real wage.

- **Plots**:

$$\begin{aligned} \frac{\partial}{\partial \lambda_w}\frac{\partial \hat{w}_t^r}{\partial \hat{z}_t} &= \frac{\phi_\pi^w[1-(\gamma+\phi_y)]}{(1+\phi_\pi^w\lambda_w)^2} < 0 \\ \frac{\partial}{\partial \lambda_w}\frac{\partial \pi_t^w}{\partial \hat{z}_t} &= \frac{1-(\gamma+\phi_y)}{(1+\phi_\pi^w\lambda_w)^2} < 0 \\ \frac{\partial}{\partial \lambda_w}\frac{\partial \pi_t}{\partial \hat{z}_t} &= \frac{(1-\phi_\pi^w)[1-(\gamma+\phi_y)]}{(1+\phi_\pi^w\lambda_w)^2} > 0 \end{aligned}$$

As λ_w increases, π_t (and thus p_t) decreases by less, π_t^w (and thus w_t) decreases by more, and w_t^r increases by less.



- **Implications for** σ_z^2: If the real wage increases in response to a positive sentiment shock, but by less if λ_w increases, then firm production is characterized by less substitutability, and the volatility of output should be higher.

7.19.2 Role of CB's response to wage inflation (ϕ_π^w)

- **Summary**: A strong high ϕ_π^w implies that only a small decrease in wage inflation is required to trigger a fall in the nominal interest rate. A strong response to wage inflation also implies that an increase in expected inflation (fall in the current price level) is not required for the real interest rate to decrease to a level such that a positive sentiment shock is fulfilled. A positive sentiment shock is fulfilled as follows: The real interest rate falls in one of two wages: either the nominal interest rate falls and/or expected price inflation increases (current price level falls). By the NKPC for wage inflation, in order for wage inflation to fall when aggregate demand rises, the real wage must increase. In order for the real wage to increase when wage inflation falls, price inflation must fall by more (i.e. the current price level must fall by more than the nominal wage).

- $\phi_\pi^w = 0$:

$$\begin{aligned}\hat{w}_t^r &= (\gamma + \phi_y)\hat{z}_t \\ \pi_t^w &= \lambda_w[1 - (\gamma + \phi_y)]\hat{z}_t \\ \pi_t &= [(\lambda_w + 1)(1 - [\gamma + \phi_y]) - 1]\hat{z}_t + \hat{w}_{t-1}^r\end{aligned}$$

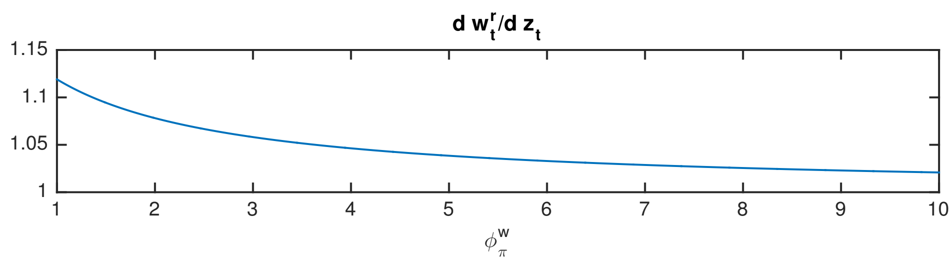
- $\phi_\pi^w \to \infty$:

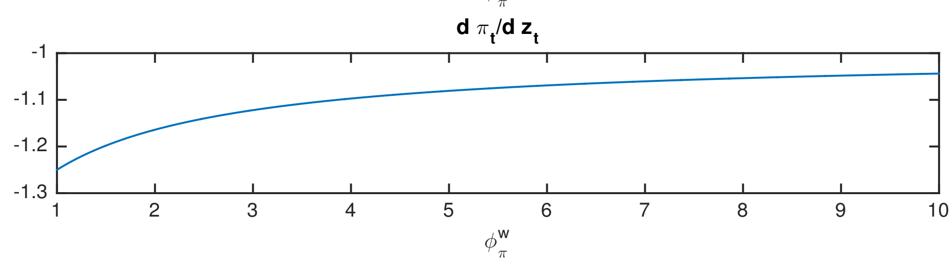
$$\begin{aligned}\hat{w}_t^r &\to \hat{z}_t \\ \pi_t^w &\to 0 \\ \pi_t &\to -\hat{z}_t + \hat{w}_{t-1}^r\end{aligned}$$

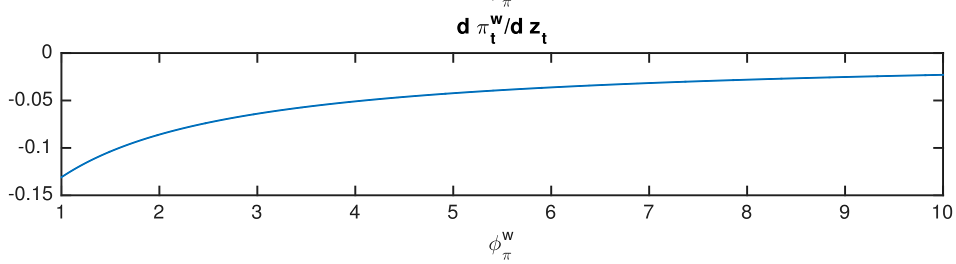
- **Plots**:

$$\begin{aligned}\frac{\partial}{\partial \phi_\pi^w}\frac{\partial \hat{w}_t^r}{\partial \hat{z}_t} &= \frac{\lambda_w[1 - (\gamma + \phi_y)]}{(1 + \phi_\pi^w \lambda_w)^2} < 0 \\ \frac{\partial}{\partial \phi_\pi^w}\frac{\partial \pi_t^w}{\partial \hat{z}_t} &= \frac{-\lambda_w^2[1 - (\gamma + \phi_y)]}{(1 + \phi_\pi^w \lambda_w)^2} > 0 \\ \frac{\partial}{\partial \phi_\pi^w}\frac{\partial \pi_t}{\partial \hat{z}_t} &= \frac{-\lambda_w(\lambda_w + 1)[1 - (\gamma + \phi_y)]}{(1 + \phi_\pi^w \lambda_w)^2} > 0\end{aligned}$$

As ϕ_π^w increases, π_t (and thus p_t) decreases by less, π_t^w (and thus w_t) decreases by less, and w_t^r increases by less.

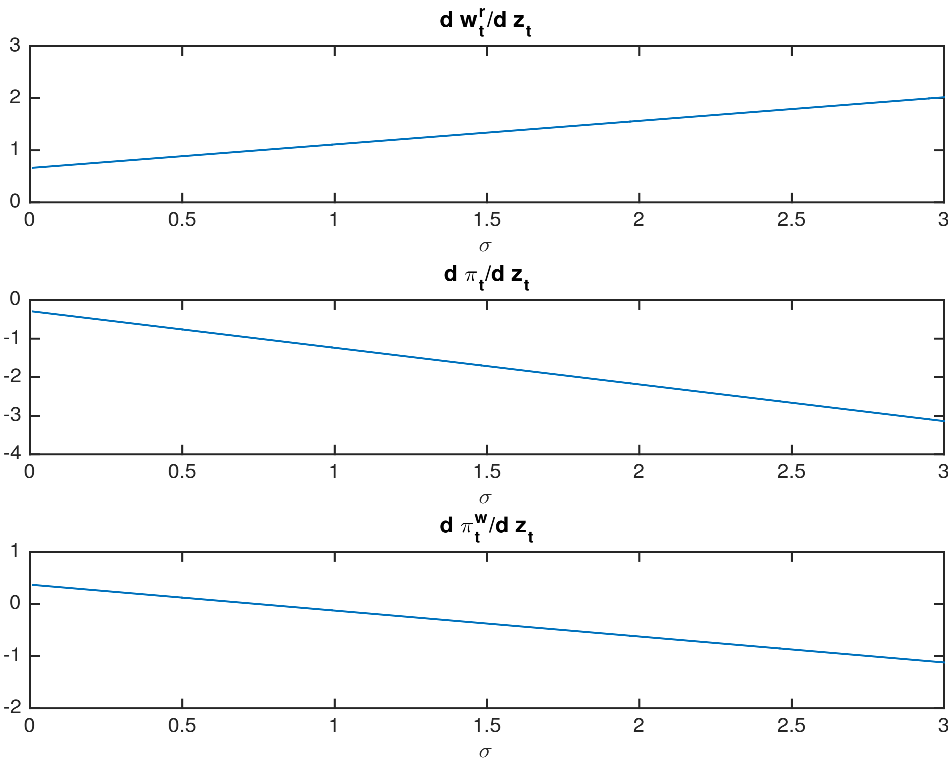






7.19.3 Role of substitution versus wealth effect on consumption, from a change in the real interest rate (γ)

- A decrease in the real interest rate has two opposing effects on consumption. The *substitution effect*: as the real interest rate falls, consumption increases as the return from savings offers lower utility than additional consumption. Consumption and savings are substitutes, and as the return from savings decreases, consumption increases. The *wealth effect* refers to a less known dynamic: as the real interest rate falls, the reduced return on savings decreases. As a result of this fall in the return to savings, households consume less.
- When γ is sufficiently small, the wealth effect dominates. From the households' optimal inter-temporal consumption decision (119), a decrease in γ renders the real interest rate more effective in changing consumption



For γ low, a smaller fall in the real interest rate is required to increase consumption on the household side. Thus, in a self-fulfilling equilibrium, wage inflation does not need to fall by as much. The result is that the real wage increases

The central bank's response to inflation or output affects the firms' optimal response to aggregate output. A central bank that responds strongly to deviations of inflation from steady state will not require a large decrease in inflation in order to lower the interest rate. Therefore, for a positive sentiment shock to be fulfilled, inflation does not need to fall by much. For firms, the consequence is that the real wage does not rise as much, which decreases the degree of substitutability in firm production.

The NKPC shows how wage inflation results from optimal wage setting by households that are able to reset wages in a given period.

$$\pi_t^w = -\lambda_w(\hat{w}_t^r - \hat{z}_t)$$

Here, we see that in order for wage inflation to fall when households consume more, the real wage must increase. *How does ϕ_π^w affect this relationship?* A central bank that targets wage inflation strongly will react to a small decrease in wage inflation by lowering the interest rate, thereby increasing consumption demand and allowing the sentiment to be fulfilled. As a smaller degree of wage inflation is necessary for this equilibrium, the real wage will not need to increase as much. Therefore, a higher ϕ_π^w reduces the substitutability in firm production.

Conditional on signal $s_{j,t} = \lambda\hat{\varepsilon}_{j,t} + (1-\lambda)\hat{z}_t$, the firms' best response under Bayesian updating is:

$$\hat{y}_{j,t} = \frac{\lambda\sigma_\varepsilon^2 + (1-\lambda)(1-\theta A)\sigma_z^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2} s_{j,t} \tag{127}$$

$$= \frac{\lambda\sigma_\varepsilon^2 + (1-\lambda)(1-\theta A)\sigma_z^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2}(\lambda\hat{\varepsilon}_{j,t} + (1-\lambda)\hat{z}_t) \tag{128}$$

Summing across firms, aggregate supply is

$$\hat{y}_t = \frac{\lambda\sigma_\varepsilon^2 + (1-\lambda)(1-\theta A)\sigma_z^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2}(1-\lambda)\hat{z}_t$$

In equilibrium, beliefs about aggregate demand are correct ($\hat{y}_t = \hat{z}_t$), which implies

$$\frac{\lambda\sigma_\varepsilon^2 + (1-\lambda)(1-\theta A)\sigma_z^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2}(1-\lambda) = 1 \tag{129}$$

In a rational expectations equilibrium, this change in the firms' optimal response to sentiments has implications for the volatility of sentiments. From (129), the volatility of sentiments and hence output is determined by structural parameters. Note that B is equivalent to the volatility of sentiments in the benchmark model.

$$\sigma_z^2 = \frac{1}{A}\frac{\lambda(1-2\lambda)}{(1-\lambda)^2\theta}\sigma_\varepsilon^2 \tag{130}$$

$$= \left[\frac{1+\phi_\pi^w\lambda_w}{\phi_\pi^w\lambda_w + (\gamma+\phi_y)}\right]\underbrace{\frac{\lambda(1-2\lambda)}{(1-\lambda)^2\theta}\sigma_\varepsilon^2}_{B} \tag{131}$$

Note that as wage flexibility (λ_w) or the central bank's response to wage inflation (ϕ_π^w) approaches infinity, A approaches 1, and the volatility of sentiments approaches B, its counterpart in the benchmark model. If wages are completely rigid ($\lambda_w = 0$) or if the central bank does not respond to wage inflation[16] ($\phi^w_\pi = 0$), then $A = \gamma + \phi_y$ and there are still sentiment driven fluctuations, as σ^2_z is given by

$$\sigma^2_z = \frac{1}{\gamma + \phi_y} \frac{\lambda(1-2\lambda)}{(1-\lambda)^2\theta} \sigma^2_\varepsilon$$

For intuition for why a higher response to wage inflation would imply a higher equilibrium volatility of sentiments, consider the response of individual firms. Relative to the baseline, firms will increase their output by more in response to a sentiment shock. Now, in order for sentiments to be self-fulfilling, the volatility of sentiments must be high so that firms attribute most of the signal to sentiments, and reduce their output. To see this more clearly, decompose the firm's response (128) in the imperfect information case:

$$\hat{y}_{j,t} = \underbrace{\frac{\lambda\sigma^2_\varepsilon}{\lambda^2\sigma^2_\varepsilon + (1-\lambda)^2\sigma^2_z} s_{j,t}}_{\mathbb{E}[\varepsilon_{j,t}|s_{j,t}]} + \left(1 - \theta \frac{\phi^w_\pi \lambda_w + (\gamma + \phi_y)}{1 + \phi^w_\pi \lambda_w}\right) \underbrace{\frac{(1-\lambda)\sigma^2_z}{\lambda^2\sigma^2_\varepsilon + (1-\lambda)^2\sigma^2_z} s_{j,t}}_{\mathbb{E}[z_t|s_{j,t}]}$$

In a rational expectations equilibrium,

$$\frac{\partial y_t}{\partial z_t} = \frac{\partial(\int_0^1 y_{j,t} dj)}{\partial z_t}$$

On the left hand side, as aggregate demand moves one-for-one with sentiments ($\hat{y}_t = \hat{z}_t$), aggregate supply must do so as well. If a central bank that targets inflation reduces substitutability of firm output, thereby increasing a firm's response to sentiments, then in equilibrium, there must be a corresponding increase in the volatility of sentiments for firms to reduce their output in response to the sentiment component of their signal ("pass-through of y_t on $\mathbb{E}[y_t]$"). This is because an increase in σ^2_z leads firms to attribute more of their signal to z_t.

$$\hat{y}_t = \underbrace{\frac{\lambda\sigma^2_\varepsilon}{\lambda^2\sigma^2_\varepsilon + (1-\lambda)^2\sigma^2_z} (1-\lambda)\hat{z}_t}_{\text{pass-through of } y_t \text{ on } \mathbb{E}[\varepsilon_{j,t}]} + \left(1 - \theta \frac{\phi^w_\pi \lambda_w + (\gamma + \phi_y)}{1 + \phi^w_\pi \lambda_w}\right) \underbrace{\frac{(1-\lambda)\sigma^2_z}{\lambda^2\sigma^2_\varepsilon + (1-\lambda)^2\sigma^2_z} (1-\lambda)\hat{z}_t}_{\text{pass-through of } y_t \text{ on } \mathbb{E}[y_t]}$$

In equilibrium, $\hat{y}_t = \hat{z}_t$, so it follows that

$$\sigma^2_y = \sigma^2_z$$

A central bank whose goal is to stabilize output and inflation faces tradeoffs. Equation (123) can be used to derive a relationship between the volatility of inflation and the volatility of output:

$$\sigma^2_{\pi^w} = \left(\frac{\lambda_w(1 - [\gamma + \phi_y])}{1 + \phi^w_\pi \lambda_w}\right)^2 \sigma^2_y$$

[16] In contrast, when the central bank does not respond to price inflation ($\phi_\pi = 0$) in the case with wages set one period in advance, $\sigma^2_z = 0$.

As $\hat{y}_t = \hat{z}_t$ in equilibrium, $\sigma_y^2 = \sigma_z^2$, where σ_z^2, from (131), is given by

$$\sigma_z^2 = \left[\frac{1+\phi_\pi^w \lambda_w}{\phi_\pi^w \lambda_w + (\gamma+\phi_y)}\right] \frac{\lambda(1-2\lambda)}{(1-\lambda)^2\theta}\sigma_\varepsilon^2$$

Replacing σ_z^2,

$$\sigma_{\pi^w}^2 = \frac{(\lambda_w(1-[\gamma+\phi_y]))^2}{(1+\phi_\pi^w\lambda_w)(\phi_\pi^w\lambda_w+[\gamma+\phi_y])}\frac{\lambda(1-2\lambda)}{(1-\lambda)^2\theta}\sigma_\varepsilon^2$$

As the central bank increases its response to wage inflation (ϕ_π^w), the volatility of wage inflation declines, but this comes at the expense of higher volatility of output. Assuming $\gamma+\phi_y > 1$, $\frac{\partial \sigma_z^2}{\partial \phi_\pi^w} > 0$

$$\frac{\partial \sigma_z^2}{\partial \phi_\pi^w} = \frac{\lambda_w(\gamma+\phi_y-1)}{(\phi_\pi^w\lambda_w+\gamma+\phi_y)^2}\frac{\lambda(1-2\lambda)}{(1-\lambda)^2\theta}\sigma_\varepsilon^2$$

Conversely, the more the central bank responds to output, the less volatile output becomes, but the more volatile wage inflation is in equilibrium. Assuming again that $\gamma+\phi_y > 1$, $\frac{\partial \sigma_{\pi^w}^2}{\partial \phi_y} > 0$

$$\frac{\partial \sigma_{\pi^w}^2}{\partial \phi_y} = \frac{\lambda_w^2(\gamma+\phi_y-1)(\gamma+\phi_y+1+2\phi_\pi^w\lambda_w)}{(\phi_\pi^w\lambda_w+\gamma+\phi_y)^2}\frac{1}{1+\phi_\pi^w\lambda_w}\frac{\lambda(1-2\lambda)}{(1-\lambda)^2\theta}\sigma_\varepsilon^2$$

One observation that can be made from the results in this section is that σ_z^2 is now bounded below by $\frac{1}{\gamma+\phi_y}\sigma_\varepsilon^2$ if $\lambda_w = 0$ or $\phi_\pi^w = 0$, whereas in the case where all households set wages one period in advance, $\sigma_z^2 \in (0,\infty)$. This result stems from the fact that A (see equation 125), which represents the degree to which the central bank's response, together with the optimal wage setting and inter-temporal consumption decisions of households, affects the equilibrium price level, and hence the firms' response to sentiment shocks. As A is bounded now[17], taking on values between $(1, \gamma+\phi_y)$, the degree of strategic substitutability that characterizes firm production is bounded.

This boundedness of the strategic substitutability in firm production reflects the difference in how a sentiment shock is fulfilled in this model. For a positive sentiment to be fulfilled, the real interest must fall, which can happen in one of two ways: (1) either the nominal interest rate falls (which requires a decrease in wage inflation) or (2) expected price inflation ($\mathbb{E}_t\pi_{t+1}$) falls. Unlike in the case where all households set wages one period in advance, $\mathbb{E}_t\pi_{t+1}$ is no longer equal to zero, but equal to the real wage (124). Note that by the New Keynesian Phillips curve for wage inflation, for wage inflation to fall when there is a positive *iid* sentiment shock, the real wage must increase.

$$\pi_t^w = \beta\mathbb{E}_t\pi_{t+1}^w - \lambda_w(\hat{w}_t^r - \hat{c}_t)$$

[17] In the case where all households set wages one period in advance, $A = \frac{\gamma+\phi_y}{\phi_\pi} \in (0,\infty)$

Since an increase in the real wage, through its effect on $\mathbb{E}_t \pi_{t+1}$, allows a positive sentiment shock to be fulfilled, there is an upper bound on the real wage in equilibrium, a upper bound on the degree of substitutability in firms' production, which results in a lower bound on the volatility of sentiments.

From (121), it is evident that a positive sentiment shock always increases the real wage. A very strong response to wage inflation (ϕ_π^w) can mitigate the degree to which the real wage will increase to make a positive sentiment shock self-fulfilling, but this increase is essential. In other words, for a nominal interest rate decrease to fulfill a positive sentiment shock, wage inflation must decrease. However, wage inflation can not decrease with a positive sentiment shock, unless the real wage increases, which explains the lower bound on the real wage in equilibrium, the lower bound on the degree of substitutability in firms' production, and hence the upper bound on the volatility of sentiments.